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GAUGING OF GEOMETRIC ACTIONS AND INTEGRABLE HIERARCHIES OF THE KP TYPE

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This work consists of two interrelated parts. First, we derive massive gauge-invariant generalizations of geometric actions on coadjoint orbits of arbitrary (infinite-dimensional) groups G with central extensions, with gauge group H being certain (infinite-dimensional) subgroup of G. We show that there exist generalized "zero-curvature" representations of the pertinent equations of motion on the coadjoint orbit. Second, in the special case of G being the Kac–Moody group, the equations of motion of the underlying gauged WZNW geometric action are identified as additional-symmetry flows of generalized Drinfeld–Sokolov integrable hierarchies based on the loop algebra $\widehat{\mathcal{G}}$. For $\widehat{\mathcal{G}} = \widehat{\mathrm{SL}}(M+R)$ the latter hierarchies are equivalent to the class $c\mathrm{KP}_{R,M}$ of constrained (reduced) KP hierarchies. We describe in some detail the loop algebras of additional (nonisospectral) symmetries of $c\mathrm{KP}_{R,M}$ hierarchies. Apart from gauged WZNW models, certain higher-dimensional nonlinear systems such as Davey–Stewartson and N-wave resonant systems are also identified as additional symmetry flows of $c\mathrm{KP}_{R,M}$ hierarchies. Also we present the explicit derivation of the general Darboux–Bäcklund solutions of $c\mathrm{KP}_{R,M}$ hierarchies preserving their additional (nonisospectral) symmetries, which for $R = 1$ contain among themselves solutions to the gauged $\mathrm{SL}(M+1)/\mathrm{U}(1) \times \mathrm{SL}(M)$ WZNW field equations.

1. Introduction

The Wess–Zumino–Novikov–Witten (WZNW) models[1] originally appeared in the literature as effective actions incorporating the anomalies in gauge theories with quantized chiral fermionic matter fields. Subsequently, with the advent of string theory WZNW models and their gauged versions[2] (embracing a large class of $D = 2$ conformal field theories which provide the Lagrangian description of Goddard–Kent–Olive coset construction[3]) have been recognized as fundamental building blocks of the latter describing various ground states of string dynamics.

Standard WZNW models and their gauged versions are themselves special cases (when the dynamical fields take values in the Kac–Moody groups) of a much broader class of geometric dynamical (Hamiltonian) models on coadjoint orbits of arbitrary (infinite-dimensional) groups with central extensions. These coadjoint-orbit models can also be viewed as anomalous quantum effective actions of matter fields with gauged (infinite-dimensional) Noether symmetry groups. Moreover, all basic properties of standard (gauged) WZNW models, in particular, the fundamental Polyakov–Wiegmann group composition law,[4] have their natural extensions[5,6] in the general setting of geometric coadjoint-orbit models. Nontrivial examples of such more general geometric models include the effective action of induced $D = 2$ gravity[7–9] and its (extended) supersymmetric generalizations;[10,5] the effective WZNW-type action of the toroidal membrane;[11,12] the effective WZNW-type action of induced $\mathbf{W}_{\mathbf{1}+\infty}$-gravity,[13] WZNW models based on two-loop Kac–Moody algebras,[14] etc. (see Sec. 3 below).

In the framework of the general formalism for geometric actions on coadjoint orbits of (infinite-dimensional) groups with central extensions proposed in Ref. 5 each physical model is fully characterized by few fundamental ingredients: (i) pairing $\langle\cdot|\cdot\rangle$ between Lie algebra $\mathcal{G}$ and its dual $\mathcal{G}^*$; (ii) (non)trivial Lie algebra two-cocycle $\omega(\cdot,\cdot)$ yielding the central extension and $\mathcal{G}^*$-valued group one-cocycle associated with the latter; (iii) the fundamental $\mathcal{G}$-valued Maurer–Cartan form on G. These ingredients carry the whole information about the symmetry structure of the models in question and enter into the general recipe for constructing the pertinent coadjoint-orbit geometric action. Classical r-matrices and Yang–Baxter equations appear naturally in this geometric setting.[15,10]

The standard (gauged) WZNW models are integrable in a sense that their pertinent equations of motion admit "zero-curvature" Lax representation.[16] In certain cases gauged WZNW field equations can be shown to belong to a whole integrable hierarchy of infinitely many soliton-like nonlinear evolution equations (i.e. integrability in a strong sense). Namely, under certain conditions the equations of motion of gauged WZNW models can be identified as additional (nonisospectral) symmetry flows of KP-type integrable hierarchies (or generalized Drinfeld–Sokolov hierarchies from algebraic point of view; see Sec. 7 below). These properties justify the natural question whether one can extend the notion of integrability or "zero-curvature" Lax representation of ordinary gauged WZNW models to the general case of gauged geometric actions on coadjoint orbits of arbitrary (infinite-dimensional) groups with central extensions. The affirmative answer to this question is provided in Sec. 5 below.

Going back to the identification of ordinary gauged WZNW field equations as additional symmetry flows of KP-type integrable hierarchies, one observes that now it is possible to employ the standard Darboux–Bäcklund techniques from Sato pseudo-differential operator approach to KP-type hierarchies for a systematic generation of soliton-like solutions of gauged WZNW equations of motion.

To this end let us recall that Kadomtsev–Petriashvili (KP) hierarchy of integrable soliton evolution equations, together with its reductions and multi-component (matrix) generalizations, describe a variety of physically important nonlinear phenomena (for a review, see e.g. Refs. 17 and 18). Constrained (reduced) KP models are intimately connected with the matrix models in nonperturbative string theory of elementary particles at ultrahigh energies (Ref. 19 and references therein). They provide an unified description of a number of basic soliton equations such as (modified) Korteweg–de-Vries, nonlinear Schrödinger (AKNS hierarchy in general), Yajima–Oikawa, coupled Boussinesq-type equations etc. Dispersionless limits of KP models were recently found[20] to play a fundamental role in the description of interface dynamics (the so-called Laplacian growth problem). Furthermore, multi-component (matrix) KP hierarchies are known to contain such physically interesting systems as two-dimensional Toda lattice, Davey–Stewartson, N-wave resonant system etc. It has been shown recently in Ref. 21 that multicomponent KP tau-functions provide solutions to the basic Witten–Dijkgraaf–Verlinde–Verlinde equations in topological field theory.

Multicomponent (matrix) KP hierarchies can be identified as ordinary (scalar) one-component KP hierarchies supplemented with a special set of commuting additional symmetry flows, namely, the Cartan subalgebra of the underlying loop algebra of additional symmetries. This construction was initially proposed in Refs. 42 and it is further elaborated on in Subsec. 6.4 below. In particular, Davey–Stewartson[44] and N-wave resonant systems are shown to arise as symmetry flows of ordinary $c\mathrm{KP}_{R,M}$ hierarchies (see Subsec. 6.5 below). The above identification leads to new systematic methods of constructing soliton-like solutions of multi-component KP hierarchies by employing the well-established techniques of Darboux–Bäcklund transformations in ordinary one-component KP hierarchies. These issues are discussed in detail in Sec. 8. We present there the explicit construction of Darboux–Bäcklund orbits of solutions for the tau-function of $c\mathrm{KP}_{R,M}$ hierarchies which simultaneously preserve the additional (nonisospectral) symmetries of the latter. The pertinent tau-functions are given in terms of generalized Wronskian-like determinants which contain among themselves solutions of the $\mathrm{SL}(M+1)/\mathrm{U}(1)\times\mathrm{SL}(M)$ gauged WZNW field equations. A subclass of the generalized Wronskian-like determinant solutions appears in the form of *multiple Wronskians* which contain as special cases the well-known (multi-)dromion solutions[22,23] of Davey–Stewartson equations.

The plan of exposition in the present paper is as follows. In Sec. 2 we briefly recapitulate the main ingredients of the general formalism[5] for construction of geometric actions on coadjoint orbits of arbitrary (infinite-dimensional) groups with central extensions. Section 3 contains various physically interesting non-trivial examples of such geometric coadjoint-orbit actions generalizing the ordinary WZNW actions. In Sec. 4 we describe the general procedure for gauging of arbitrary coadjoint-orbit geometric actions generalizing the usual procedure for gauging of WZNW models. Section 5 provides the "zero-curvature" representation of the equations of motion of gauged geometric actions on arbitrary group coadjoint orbits. Section 6 is devoted, after a brief review of Sato pseudo-differential operator formulation of KP-type integrable hierarchies, to the construction of the full-loop algebra of additional (nonisospectral) symmetries for constrained (reduced) KP hierarchies.

In particular, multicomponent (matrix) KP hierarchies are obtained out of ordinary one-component (scalar) KP hierarchies endowed with appropriate infinite sets of commuting additional symmetry flows. In Sec. 7 it is first shown that gauged WZNW equations of motion can be viewed as additional (nonisospectral) symmetry flows of generalized Drinfeld–Sokolov hierarchies containing among themselves the class of $c\mathrm{KP}_{R,M}$ constrained KP hierarchies. Further, in the case of $c\mathrm{KP}_{R,M}$ hierarchies we explicitly demonstrate the interrelation between Sato pseudo-differential operator and algebraic (generalized) Drinfeld–Sokolov formulations. Section 8 contains a detailed systematic construction of Darboux–Bäcklund solutions for the tau-functions of all $c\mathrm{KP}_{R,M}$ hierarchies preserving their additional symmetries. As a byproduct one obtains in this way solutions for the equations of motion of gauged $\mathrm{SL}(M+1)/\mathrm{U}(1) \times \mathrm{SL}(M)$ WZNW models.

The results of Sec. 6 have been previously reported in a short form in Ref. 24.

2. General Formalism for Geometric Actions on Coadjoint Orbits

2.1. *Basic ingredients*

Consider arbitrary (infinite-dimensional) group G with a Lie algebra $\mathcal{G}$ and its dual space $\mathcal{G}^*$. The adjoint and coadjoint actions of G and $\mathcal{G}$ on $\mathcal{G}$ and $\mathcal{G}^*$ are given by

$$Ad(g)(X) = gXg^{-1}\,, \qquad ad(X_1)X_2 = [X_1, X_2]\,, \tag{1}$$

$$\langle Ad^*(g)U|X\rangle = \langle U|Ad(g^{-1})X\rangle\,, \quad \langle ad^*(X_1)U|X_2\rangle = -\langle U|ad(X_1)X_2\rangle\,. \tag{2}$$

Here $g \in G$ and X, $X_{1,2} \in \mathcal{G}$, $U \in \mathcal{G}^*$ are arbitrary elements, whereas $\langle\cdot|\cdot\rangle$ indicates the natural bilinear form "pairing" $\mathcal{G}$ and $\mathcal{G}^*$.

Our primary interest is in infinite-dimensional Lie algebras with a central extension $\tilde{\mathcal{G}} = \mathcal{G} \oplus \mathbb{R}$ of $\mathcal{G}$ and, correspondingly, an extension $\tilde{\mathcal{G}}^* = \mathcal{G}^* \oplus \mathbb{R}$ of the dual space $\mathcal{G}^*$. The central extension is given by a linear operator $\hat{s} : \mathcal{G} \to \mathcal{G}^*$ satisfying:

$$\hat{s}([X_1, X_2]) = ad^*(X_1)\hat{s}(X_2) - ad^*(X_2)\hat{s}(X_1)\,, \tag{3}$$

which defines a nontrivial two-cocycle on the Lie algebra $\mathcal{G}$:

$$\omega(X_1, X_2) \equiv -\lambda\langle\hat{s}(X_1)|X_2\rangle \quad \text{for } \forall\ X_{1,2} \in \mathcal{G}\,, \tag{4}$$

where λ is a numerical normalization constant. The Jacobi identity (3) can be integrated ($X_2 \to g = \exp X_2$) to get a unique nontrivial $\mathcal{G}^*$-valued group one-cocycle $S(g)$ in terms of the Lie-algebra cocycle operator $\hat{s}$ (provided $H^1(G) = \emptyset$, $\dim H^2(G) = 1$; see Ref. 25):

$$ad^*(X)S(g) = Ad^*(g)\hat{s}(Ad(g^{-1})X) - \hat{s}(X) \quad \text{for } \forall\ X \in \mathcal{G}\,, \tag{5}$$

satisfying the relations:

$$\hat{s}(X) = \frac{d}{dt}S(e^{tX})\Big|_{t=0}\,, \qquad S(g_1 g_2) = S(g_1) + Ad^*(g_1)S(g_2)\,. \tag{6}$$

One can easily generalize the adjoint and coadjoint actions of G and $\mathcal{G}$ to the case with a central extension (acting on elements (X,n), $(X_{1,2},n_{1,2}) \in \tilde{\mathcal{G}}$ and $(U,c) \in \tilde{\mathcal{G}}^*$; see e.g. Ref. 5):

$$\tilde{A}d(g)(X,n) = (Ad(g)X, n + \lambda\langle S(g^{-1})|X\rangle)\,, \tag{7}$$

$$\tilde{a}d(X_1,n_1)(X_2,n_2) \equiv [(X_1,n_1),(X_2,n_2)] = (ad(X_1)X_2, -\lambda\langle \hat{s}(X_1)|X_2\rangle)\,, \tag{8}$$

$$\begin{aligned} \tilde{A}d^*(g)(U,c) &= (Ad^*(g)U + c\lambda S(g), c)\,, \\ \tilde{a}d^*(X,n)(U,c) &= (ad^*(X)U + c\lambda \hat{s}(X), 0)\,. \end{aligned} \tag{9}$$

Also, the bilinear form $\langle\cdot|\cdot\rangle$ on $\mathcal{G}^* \otimes \mathcal{G}$ can be extended to a bilinear form on $\tilde{\mathcal{G}}^* \otimes \tilde{\mathcal{G}}$ as:

$$\langle (U,c)|(\xi,n)\rangle = \langle U|\xi\rangle + cn\,. \tag{10}$$

From physical point of view the interpretation of the $\mathcal{G}$-cocycle $\hat{s}$ is that of "anomaly" of the Lie algebra (i.e. existence of a c-number term in the commutator (8)), whereas the group cocycle $S(g)$ is the integrated "anomaly," i.e. the "anomaly" for finite group transformations (see Eqs. (7) and (6)).

Another basic geometric object is the fundamental $\mathcal{G}$-valued Maurer–Cartan one-form $Y(g)$ on G with values in $\mathcal{G}$ satisfying:

$$dY(g) = \frac{1}{2}[Y(g), Y(g)]\,. \tag{11}$$

It is related to the group one-cocycle $S(g)$ through the equation:

$$dS(g) = ad^*(Y(g))S(g) + \hat{s}(Y(g))\,, \tag{12}$$

and possesses group one-cocycle property similar to that of $S(g)$ (6):

$$Y(g_1 g_2) = Y(g_1) + Ad(g_1)Y(g_2)\,. \tag{13}$$

Using (5) one can rewrite relation (12) in another useful form:

$$dS(g) = -Ad^*(g)\hat{s}(Y(g^{-1}))\,. \tag{14}$$

The group- and algebra-cocycles $S(g)$ and $\hat{s}(X)$ can be generalized to include trivial (co-boundary) parts ((U_0,c) being an arbitrary point in the extended dual space $\tilde{\mathcal{G}}^*$):

$$\Sigma(g) \equiv \Sigma(g;(U_0,c)) = c\lambda S(g) + Ad^*(g)U_0 - U_0\,, \tag{15}$$

$$\hat{\sigma}(X) \equiv \hat{\sigma}(\xi;(U_0,c)) = ad^*(X)U_0 + c\lambda\hat{s}(X) = \frac{d}{dt}\Sigma(e^{t\xi})\Big|_{t=0}\,. \tag{16}$$

The generalized cocycles (15) and (16) satisfy the same relations as (6), (12) and (5).

2.2. *Coadjoint orbits*

The coadjoint orbit of G, passing through the point (U_0, c) of the dual space $\tilde{\mathcal{G}}^*$, is defined as (cf. (9)):

$$\mathcal{O}_{(U_0,c)} \equiv \{(U(g), c) \in \tilde{\mathcal{G}}^*; U(g) = U_0 + \Sigma(g) = Ad^*(g)U_0 + c\lambda S(g)\}\,. \tag{17}$$

The orbit (17) is a right coset $\mathcal{O}_{(U_0,c)} \simeq G/G_{\rm stat}$ where $G_{\rm stat}$ is the stationary subgroup of the point (U_0, c) w.r.t. the coadjoint action (9):

$$G_{\rm stat} = \{k \in G; \Sigma(k) \equiv c\lambda S(k) + Ad^*(k)U_0 - U_0 = 0\}\,. \tag{18}$$

The Lie algebra corresponding to $G_{\rm stat}$ is

$$\mathcal{G}_{\rm stat} \equiv \{X_0 \in \mathcal{G}; \hat{\sigma}(X_0) \equiv ad^*(X_0)U_0 + c\lambda\hat{s}(X_0) = 0\}\,. \tag{19}$$

Now, using the basic geometric objects from Subsec. 2.1, we can express the Kirillov–Knstant symplectic form $\Omega_{\rm KK}$[26] on $\mathcal{O}_{(U_0,c)}$ for any infinite-dimensional (centrally extended) group G in a simple compact form.[5] Namely, introducing the centrally extended objects:

$$\tilde{\Sigma}(g) \equiv (\Sigma(g), c) \in \tilde{\mathcal{G}}^*\,, \qquad \tilde{Y}(g) \equiv (Y(g), m_Y(g)) \in \tilde{\mathcal{G}}\,, \tag{20}$$

$$d\tilde{\Sigma}(g) = \tilde{a}d^*(\tilde{Y}(g))\tilde{\Sigma}(g)\,, \qquad d\tilde{Y}(g) = \frac{1}{2}[\tilde{Y}(g), \tilde{Y}(g)]\,, \tag{21}$$

we obtain (using (10) and (12)):

$$\Omega_{\rm KK} = -d(\langle\tilde{\Sigma}(g)|\tilde{Y}(g)\rangle) = -\frac{1}{2}\langle d\tilde{\Sigma}(g)|\tilde{Y}(g)\rangle\,. \tag{22}$$

2.3. *Geometric actions and symmetries*

The geometric action on a coadjoint orbit $\mathcal{O}_{(U_0,c)}$ of arbitrary infinite-dimensional (centrally extended) group G can now be written down compactly as:[5,6]

$$W[g] = \int d^{-1}\Omega_{\rm KK} = -\int \langle\tilde{\Sigma}(g)|\tilde{Y}(g)\rangle\,, \tag{23}$$

or, in more detail, introducing the explicit expressions (20), (21), (15) and (16):

$$W[g] = \int \langle U_0|Y(g^{-1})\rangle - c\lambda \int \left[\langle S(g)|Y(g)\rangle - \frac{1}{2}d^{-1}(\langle\hat{s}(Y(g))|Y(g)\rangle)\right]. \tag{24}$$

The integral in (23), (24) is over one-dimensional curve on the phase space $\mathcal{O}_{(U_0,c)}$ with a "time-evolution" parameter t. Along the curve the exterior derivative becomes $d = dt\partial_t$ and the projection of the Maurer–Cartan one-form $Y(g)$ is: $Y(g) = dtY_t(g)$.

The fundamental Poisson brackets resulting from the geometric action (24) read:

$$\{\langle\tilde{\Sigma}(g)|\tilde{X}_1\rangle, \langle\tilde{\Sigma}(g)|\tilde{X}_2\rangle\}_{\rm PB} = -\langle\tilde{\Sigma}(g)|[\tilde{X}_1, \tilde{X}_2]\rangle\,, \qquad \tilde{X}_{1,2} \equiv (X_{1,2}, n_{1,2})\,, \tag{25}$$

or in the case of orbits $\mathcal{O}_{(U_0=0,c)}$ (cf. (17)):

$$\{\langle S(g)|X_1\rangle, \langle S(g)|X_2\rangle\}_{\rm PB} = -\langle S(g)|[X_1, X_2]\rangle + c\lambda\langle\hat{s}(X_1)|X_2\rangle\,. \tag{26}$$

In particular, (25) shows that $\tilde{\Sigma}(g)$ is an equivariant moment map.

Using the group cocycle properties of $S(g)$ and $Y(g)$ (Eqs. (6) and (13)) we derive the following fundamental group composition law[6] (with $\Sigma(g)$ as in (15)):

$$W[g_1 g_2] = W[g_1] + W[g_2] + \int \langle \Sigma(g_2) | Y(g_1^{-1}) \rangle \,. \tag{27}$$

Equation (27) is a generalization of the famous Polyakov–Wiegmann composition law[4] in WZNW models to geometric actions on coadjoint orbits of arbitrary (infinite-dimensional) groups with central extensions.

Equation (27) contains the whole information about the symmetries of the geometric action (24). Under arbitrary left and right infinitesimal group translations:

$$g \to (\mathbb{1} + \varepsilon_{\mathrm{L}})g \,, \quad g \to g(\mathbb{1} + \varepsilon_{\mathrm{R}}) \,, \quad \varepsilon_{\mathrm{L,R}} \in \mathcal{G} \,, \tag{28}$$

we obtain using (27):

$$\delta_L W[g] = -\int \langle \Sigma(g) | d\varepsilon_L \rangle \,, \tag{29}$$

i.e. $\Sigma(g)$ (15) is a Noether conserved current $\partial_t \Sigma(g) = 0$, and:

$$\delta_R W[g] = \int \langle \hat{\sigma}(\varepsilon_R) | Y(g^{-1}) \rangle = -\int \langle \hat{\sigma}(Y(g^{-1})) | \varepsilon_R \rangle \,. \tag{30}$$

Recalling (19) we find "gauge" invariance of $W[g]$ under right group translations from the stationary subgroup G_{stat} (18) of the orbit $\mathcal{O}_{(U_0,c)}$ (17): $\delta_R W[g] = 0$ for $\forall \varepsilon_R \in \mathcal{G}_{\mathrm{stat}}$ (19). This reveals the geometric meaning of "hidden" local symmetries[7] in models with arbitrary infinite-dimensional Noether symmetry groups.

3. Examples of Geometric Actions on Coadjoint Orbits

3.1. *The Kac–Moody groups*

The Kac–Moody group elements $g \simeq g(x)$ are smooth mappings $S^1 \to G_0$, where G_0 is a finite-dimensional Lie group with generators $\{T^A\}$. The explicit form of (7)–(9) reads in this case:

$$\begin{gathered}
Ad(g)X = g(x)X(x)g^{-1}(x) \,, \quad ad(X_1)X_2 = [X_1(x), X_2(x)] \,, \\
X_{1,2}(x) = X_{1,2}^A(x) T_A \,, \\
Ad^*(g)U = g(x)U(x)g^{-1}(x) \,, \quad ad^*(X)U = [X(x), U(x)] \,, \\
U(x) = U_A(x) T^A \,, \\
\hat{s}(X) = \partial_x X(x) \,, \quad S(g) = \partial_x g(x) g^{-1}(x) \,, \\
Y(g) = dg(x) g^{-1}(x) \,.
\end{gathered} \tag{31}$$

Plugging (31) into (24) one obtains the well-known WZNW action[1] for G_0-valued chiral fields coupled to an external "potential" $U_0(x)$, whereas Eq. (27) reduces to the Polyakov–Wiegmann group composition law for WZNW actions.[4]

3.2. *Virasoro group*

The Virasoro group elements $g \simeq F(x)$ are smooth diffeomorphisms of the circle S^1. Group multiplication is given by composition of diffeomorphisms in inverse order: $g_1 \cdot g_2 = F_2 \circ F_1(x) = F_2(F_1(x))$. Equations (7)–(9) have now the following explicit form:

$$\begin{gathered}
Ad(F)X = (\partial_x F)^{-1} X(F(x)) \,, \quad Ad^*(F)U = (\partial_x F)^2 U(F(x)) \,; \\
ad(X_1)X_2 \equiv [X_1, X_2] = X_1 \partial_x X_2 - (\partial_x X_1) X_2 \,, \\
ad^*(X)U = X\partial_x U + 2(\partial_x X)U \,; \\
\hat{s}(\xi) = \partial_x^3 \xi \,, \quad S(F) = \frac{\partial_x^3 F}{\partial_x F} - \frac{3}{2}\left(\frac{\partial_x^2 F}{\partial_x F}\right)^2 \,, \quad Y(F) = \frac{dF}{\partial_x F} \,.
\end{gathered} \tag{32}$$

Here $S(F)$ is the well-known Schwarzian. Plugging (32) into the general expressions (24) and (27) one reproduces the well-known Polyakov $D = 2$ gravity action (coupled to an external stress-tensor $U_0(x)$):

$$W[F] = \int dt\, dx \left[-U_0(F(t,x))\partial_x F \partial_t F + \frac{c}{48\pi} \frac{\partial_t F}{\partial_x F} \left(\frac{\partial_x^3 F}{\partial_x F} - 2\frac{(\partial_x^2 F)^2}{(\partial_x F)^2} \right) \right] \tag{33}$$

and its group composition law.[7,8]

3.3. $(N,0)$ *super-Virasoro group* $(N \leq 4)$

Here we shall use the manifestly $(N,0)$ supersymmetric formalism. The points of the $(N,0)$ superspace are labeled as (t,z), $z \equiv (x, \theta^i)$, $i = 1, \ldots, N$. The group elements are given by superconformal diffeomorphisms:

$$z \equiv (x, \theta^j) \to \tilde{Z} \equiv (F(x, \theta^j), \tilde{\Theta}^i(x, \theta^j)) \tag{34}$$

obeying the superconformal constraints:

$$D^j F - i\tilde{\Theta}^k D^j \tilde{\Theta}_k = 0 \,, \qquad D^j \tilde{\Theta}^l D^k \tilde{\Theta}_l - \delta^{jk}[D\tilde{\Theta}]_N^2 = 0 \,, \tag{35}$$

with the following superspace notations:

$$D^i = \frac{\partial}{\partial \theta_i} + i\theta^i \partial_x \,, \quad D^N \equiv \frac{1}{N!}\epsilon_{i_1 \cdots i_N} D^{i_1} \cdots D^{i_N} \,, \quad [D\tilde{\Theta}]_N^2 \equiv \frac{1}{N} D^m \tilde{\Theta}^n D_m \tilde{\Theta}_n \,. \tag{36}$$

The $(N,0)$ supersymmetric analogs of (32) read:

$$Ad(\tilde{Z})X = ([D\tilde{\Theta}]_N^2)^{-1} X(\tilde{Z}(z)) \,, \quad Ad^*(\tilde{Z})U = ([D\tilde{\Theta}]_N^2)^{2-\frac{N}{2}} U(\tilde{Z}(z)) \,, \tag{37}$$

$$\begin{gathered}
ad(X_1)X_2 \equiv [X_1, X_2] = X_1 \partial_x X_2 - (\partial_x X_1) X_2 - \frac{i}{2} D_k X_1 D^k X_2 \,, \\
ad^*(X)U = X\partial_x U + \left(2 - \frac{N}{2}\right)(\partial_x X)U - \frac{i}{2} D_k \xi D^k U \,, \\
\hat{s}_N(X) = i^{N(N-2)} D^N \partial_x^{3-N} X \,, \quad Y_N(\tilde{Z}) = (dF + i\tilde{\Theta}^j d\tilde{\Theta}_j)([D\tilde{\Theta}]_N^2)^{-1} \,.
\end{gathered} \tag{38}$$

The associated $\mathcal{G}^*$-valued group one-cocycles $S_N(\tilde{Z})$ coincide with the well-known[27] $(N,0)$ super-Schwarzians. Inserting the latter and (38) into (24) one obtains the $(N,0)$ supersymmetric generalization of the Polyakov $D=2$ gravity action for any $N \leq 4$:[10,5]

$$\begin{aligned} W_N[\tilde{Z}] = \int dt(dz)[\partial_t(\ln[D\tilde{\Theta}]^2_N)D^N\partial_x^{1-N}([D\tilde{\Theta}]^2_N) \\ - U_0(\tilde{Z})([D\tilde{\Theta}]^2_N)^{2-\frac{N}{2}}Y_N(\tilde{Z})] \,. \end{aligned} \tag{39}$$

3.4. *Group of area-preserving diffeomorphisms on torus with central extension* $\widetilde{\mathbf{SDiff}}(T^2)$

The elements of $\widetilde{\mathrm{SDiff}}(T^2)$ are described by smooth diffeomorphisms $T^2 \ni \mathbf{x} \equiv (x^1,x^2) \to F^i(\mathbf{x}) \in T^2 (i=1,2)$, such that $\det \left\| \frac{\partial F^i}{\partial x^j} \right\| = 1$. The Lie algebra of $\widetilde{\mathrm{SDiff}}(T^2)$ reads: $[\hat{\mathcal{L}}(\mathbf{x}), \hat{\mathcal{L}}(\mathbf{y})] = -\epsilon^{ij}\partial_i\hat{\mathcal{L}}(\mathbf{x})\partial_j\delta^{(2)}(\mathbf{x}-\mathbf{y}) - a^i\partial_i\delta^{(2)}(\mathbf{x}-\mathbf{y})$, where $\mathbf{a} \equiv (a^1, a^2)$ are the "central charges."[28] The general equations (7)–(9) now specialize to:[11]

$$\begin{gathered} Ad(\mathbf{F})X = X(\mathbf{F}(\mathbf{x}))\,, \quad ad(X_1)X_2 \equiv [X_1,X_2](\mathbf{x}) = \epsilon^{ij}\partial_i X_1(\mathbf{x})\partial_j X_2(\mathbf{x})\,, \\ Ad^*(\mathbf{F})U = U(\mathbf{F}(\mathbf{x}))\,, \quad ad^*(X)U = \epsilon^{ij}\partial_i X(\mathbf{x})\partial_j U(\mathbf{x})\,, \\ \hat{s}(X) = a^i\partial_i X(\mathbf{x})\,, \quad S(\mathbf{F}) = a^i\epsilon_{ij}(F^j(\mathbf{x}) - x^j)\,, \\ Y(\mathbf{F}) = \frac{1}{2}\epsilon_{ij}F^i dF^j + d\rho(\mathbf{F})\,, \end{gathered} \tag{40}$$

where $\partial_i\rho(\mathbf{F}) = -\frac{1}{2}\left(\epsilon_{kl}F^k\partial_i F^l + \epsilon_{ij}x^j\right)$.

Plugging (40) into (24) we get the $\widetilde{\mathrm{SDiff}}(T^2)$ co-orbit geometric action:[11]

$$W_{\widetilde{\mathrm{SDiff}}(T^2)}[\mathbf{F}] = -\frac{1}{3}\int dt\, dx^2 \left(a^k\epsilon_{kl}F^l\right)\epsilon_{ij}F^i\partial_t F^j\,. \tag{41}$$

In Ref. 12 it was shown that (41) is the Wess–Zumino anomalous effective action for the toroidal membrane in the light-cone gauge.

3.5. $\mathbf{W}_{1+\infty}$*-gravity effective action*

The $\mathbf{W}_{1+\infty}$-algebra is isomorphic to the algebra of all differential operators on the circle $\mathcal{DOP}(S^1) = \{X \equiv X(x,\partial) = \sum_{i\geq 0} X_i\partial^i\}$. Accordingly, the dual space is the space of all purely pseudo-differential operators:

$$\mathcal{DOP}^*(S^1) = \left\{ U \equiv U(x,\partial) = \sum_{j\geq 1} u_j\partial^{-j} \right\}, \tag{42}$$

where the bilinear pairing is defined by

$$\langle U|X\rangle = \int dx\,\mathrm{Res}(UX)\,; \quad \mathrm{Res}\,\mathcal{A} \equiv a_{-1} \quad \text{for any } \mathcal{A} = \sum_k a_k\partial^k\,. \tag{43}$$

The corresponding Lie group $DOP(S^1)$ is defined as formal exponentiation of the Lie algebra $\mathcal{DOP}(S^1)$, where the group elements $g(x,\partial) = \exp X(x,\partial)$ are understood again in the sense of pseudo-differential operator calculus. The relevant objects from the coadjoint orbit formalism are given in this case as follows:

$$\begin{gathered}
Ad(g)X = g(x,\partial)X(x,\partial)g^{-1}(x,\partial)\,, \qquad ad(X_1)X_2 = [X_1(x,\partial), X_2(x,\partial)]\,, \\
Ad^*(g)U = (g(x,\partial)U(x,\partial)g^{-1}(x,\partial))_-\,, \qquad ad^*(X)U = [X(x,\partial), U(x,\partial)]_-\,, \\
\hat{s}(X) = -[\ln\partial, X(x,\partial)]_-\,, \quad S(g) = -([\ln\partial, X(x,\partial)]g^{-1}(x,\partial))_-\,, \\
Y(g) = dg(x,\partial)g^{-1}(x,\partial)\,.
\end{gathered} \tag{44}$$

Everywhere in (44) products are understood in the sense of pseudo-differential operator calculus, and the subscript $(-)$ indicates taking the purely pseudo-differential part.

Now, the geometric action on a coadjoint orbit of $DOP(S^1) \simeq \mathbf{W}_{1+\infty}$, which is the WZNW effective action of induced $\mathbf{W}_{1+\infty}$-gravity, is given as[13] (for brevity we suppress the indication of arguments in $g = g(x,\partial)$ etc.):

$$\begin{aligned}
W[g] &\equiv W_{DOP(S^1)}[g] \\
&= -\int dt\,dx\,\mathrm{Res}(U_0 g^{-1}\partial_t g) + \frac{c}{4\pi}\iint dx\,\mathrm{Res} \\
&\quad \times \left([\ln\partial, g]g^{-1}\partial_t g g^{-1} - \frac{1}{2}d^{-1}\{[\ln\partial, dg\,g^{-1}] \wedge (dg\,g^{-1})\}\right)\,,
\end{aligned} \tag{45}$$

whereas the group composition law reads:[13]

$$W[gh] = W[g] + W[h] + \int dt\,dx \left\{ \left(hU_0h^{-1} - \frac{c}{4\pi}[\ln\partial, h]h^{-1} \right) g^{-1}\partial_t g \right\}. \tag{46}$$

It has been shown in the first Ref. 13 that the stationary subgroup (18) of the pertinent $DOP(S^1) \simeq \mathbf{W}_{\mathbf{1}+\infty}$ coadjoint orbit is SL(∞), which thereby appears as "hidden" symmetry of the $\mathbf{W}_{\mathbf{1}+\infty}$ geometric action (45), and that the energy–momentum component T_{++} possesses SL(∞) Sugawara form.

4. Gauging of Geometric Actions

Let us now return to the general case of arbitrary (infinite-dimensional) groups G with central extensions. Henceforth, for simplicity we will consider geometric actions (24) on coadjoint orbits (17) with $U_0 = 0$[a] and also we will set the normalization constant in (24) $-c\lambda = 1$:

$$W[g] = \int \left[\langle S(g)|Y(g)\rangle - \frac{1}{2}d^{-1}(\langle\hat{s}(Y(g))|Y(g)\rangle) \right]\,, \tag{47}$$

[a]Note that the first term in (24) containing U_0 can be interpreted as coupling to an external background field.

whereupon the generalized group composition law (27) simplifies to

$$W[g_1 g_2] = W[g_1] + W[g_2] - \int \langle S(g_2) | Y(g_1^{-1}) \rangle \,. \tag{48}$$

Now, suppose that there exist two fixed elements $E_- \in \mathcal{G}$ and $\mathcal{E}_+ \in \mathcal{G}^*$ such that they define splitting (as vector spaces) of the Lie algebra $\mathcal{G}$ and its dual $\mathcal{G}^*$ with the following properties:

$$\mathcal{G} = \mathcal{H} \oplus \mathcal{M} \,, \quad \mathcal{G}^* = \mathcal{H}^* \oplus \mathcal{M}^* \,, \quad \mathcal{M}^* \equiv \{ U | \langle U | X_H \rangle = 0 \text{ for } \forall\, X_H \in \mathcal{H} \} \,, \tag{49}$$

where

$$\mathcal{H} = \mathrm{Ker}(\mu(\mathcal{E}_+)) \,, \quad E_- \in \mathcal{H} \,, \quad \hat{s}(E_-) = 0 \text{ (i.e. } E_- \in \mathcal{G}_{\mathrm{stat}} ,\ \text{cf. (19))} \,, \tag{50}$$

and $\mu(U)(\cdot)$ is the mapping:

$$\mu(U) : \mathcal{G} \to \mathcal{G}^* \,, \quad \mu(U) X = ad^*(X) U \quad \text{for } X \in \mathcal{G} \,, \quad U \in \mathcal{G}^* \,. \tag{51}$$

Properties (50) imply that $\mathcal{H}$ is a subalgebra of $\mathcal{G}$ and that

$$\mu(\mathcal{E}_+) \mathcal{G} \equiv ad^*(\mathcal{G}) \mathcal{E}_+ \subset \mathcal{M}^* \,, \qquad ad^*(\mathcal{H}) \mathcal{M}^* \subset \mathcal{M}^* \,. \tag{52}$$

In particular, the first two relations (50) show that the fixed elements E_- and $\mathcal{E}_+$ mutually "commute":

$$ad^*(E_-) \mathcal{E}_+ = 0 \,. \tag{53}$$

The first two properties (50) arise as sufficient conditions for consistency of the equations of motion of the gauged geometric actions given below. The third property (50) arises as sufficient condition for validity of the generalized "zero-curvature" representation on the group coadjoint orbit of the equations of motion of the gauged geometric actions.

In the special case of G being Kac–Moody group (cf. Subsec. 3.1 above) where Lie algebras and dual spaces are identified, relations (49)–(50) acquire the following meaning:

$$\mathcal{G} = \mathcal{H} \oplus \mathcal{M} \,, \qquad \mathcal{H} = \mathrm{Ker}(ad(E_+)) \,, \tag{54}$$

where $\mathcal{E}_+ \equiv E_+$ and E_- are two mutually commuting fixed Kac–Moody algebra elements belonging to $\mathcal{H}$. In order to make contact with integrable models, one requires in addition to (54) that E_+ is semisimple element, i.e.:

$$\mathcal{G} = \mathcal{H} \oplus \mathcal{M} \,, \quad \mathcal{H} = \mathrm{Ker}(ad(E_+)) \,, \quad \mathcal{M} = \mathrm{Im}(ad(E_+)) \to [\mathcal{H}, \mathcal{M}] \subset \mathcal{M} \,. \tag{55}$$

Going back to the general case of coadjoint orbits of arbitrary (infinite-dimensional) groups with central extensions, let us consider "gauge" fields — the one-form $\mathcal{A}_+ \in \mathcal{H}$ and $A_- \in \mathcal{H}^*$ which are parametrized in terms of the group elements h_L, $h_R \in H$ as:

$$\mathcal{A}_+ = Y(h_L) \,, \qquad A_- = S(h_R^{-1}) \,, \tag{56}$$

where $Y(\cdot)$ and $S(\cdot)$ are the fundamental Maurer–Cartan form (12)–(13) and the nontrivial group cocycle (5) restricted on the subgroup H.

Now we are ready to introduce the following new geometric action which is a "massive" gauge-invariant generalization of (47):

$$W[g,A_+,A_-] \equiv W[h_L^{-1} g h_R^{-1}] - W[h_L^{-1} h_R^{-1}] + \int dt \langle Ad^*(h_R g^{-1} h_L)\mathcal{E}_+ | E_-\rangle \tag{57}$$

$$\begin{aligned} &= \int \left[\langle S(g)|Y(g)\rangle - \frac{1}{2} d^{-1}(\langle \hat{s}(Y(g))|Y(g)\rangle)\right] \\ &\quad - \int [\langle S(g)|\mathcal{A}_+\rangle + \langle A_-|Y(g^{-1})\rangle] \\ &\quad - \int [\langle Ad^*(g)A_-|\mathcal{A}_+\rangle - \langle A_-|\mathcal{A}_+\rangle] \\ &\quad - \int dt \langle Ad^*(h_R g^{-1} h_L)\mathcal{E}_+|E_-\rangle\,, \end{aligned} \tag{58}$$

where $\mathcal{A}_+$, A_- are as in (56). Along the curve of "time" integration in (58) the Maurer–Cartan $\mathcal{H}$-valued one-form $\mathcal{A}_+ = Y(h_L)$ becomes $\mathcal{A}_+ = A_+ dt$ similar to the fundamental $\mathcal{G}$-valued Maurer–Cartan form $Y(g) = Y_t(g)dt$ as pointed out above. Let us stress that, although the gauged geometric action (58) formally resembles ordinary G/H gauged WZNW models where G, H are Kac–Moody groups, the action (58) is valid in the much more general setting of arbitrary (infinite-dimensional) groups with central extensions (cf. the examples in Sec. 3).

The action (58) exhibits manifest "vector-like" gauge invariance under:

$$g \to h^{-1} g h\,, \quad h_L \to h^{-1} h_L\,, \quad h_R \to h_R h\,, \quad h \in H\,. \tag{59}$$

In particular, we note that the last "mass" term on the r.h.s. of (58) is gauge-invariant by itself. Also, let us emphasize that the second explicit form (58) is obtained from the first defining form (57) by using the general group composition law identities (48). The gauged geometric action (58) is a generalization of the well-known gauged WZNW actions[2] to the case of arbitrary (infinite-dimensional) groups with central extensions (see the examples in Sec. 3 above).

To derive the equations of motion for (58) we need the infinitesimal forms of the group cocycle properties (6) and (13) under left and right infinitesimal group translations (28):

$$\begin{aligned} \delta_L Y(g) &= d\varepsilon_L - ad(Y(g))\varepsilon_L\,, \\ \delta_L S(g) &= \hat{s}(\varepsilon_L) + ad^*(\varepsilon_L)S(g)\,, \\ \delta_R S(g^{-1}) &= -\hat{s}(\varepsilon_R) - ad^*(\varepsilon_R)S(g^{-1})\,. \end{aligned} \tag{60}$$

Using Eqs. (60) the variations of (58) w.r.t. $\varepsilon_L = \delta g\, g^{-1} \in \mathcal{G}$, $\varepsilon_L = \delta h_L\, h_L^{-1} \in \mathcal{H}$ and $\varepsilon_R = h_R^{-1}\delta h_R \in \mathcal{H}$ yield the following gauge-invariant equations of motion:

$$\begin{aligned} &(\partial_t - ad^*(A_+))(S(g) + Ad^*(g)A_- - A_-) + ad^*(Ad(g h_R^{-1})E_-)(Ad^*(h_L)\mathcal{E}_+) \\ &\quad + \partial_t A_- - \hat{s}(A_+) - ad^*(A_+)A_- = 0\,; \end{aligned} \tag{61}$$

$$(\partial_t - ad^*(A_+))(S(g) + Ad^*(g)A_- - A_-)|_{\mathcal{H}^*}$$
$$+ ad^*(Ad(gh_R^{-1})E_-)(Ad^*(h_L)\mathcal{E}_+)|_{\mathcal{H}^*} = 0\,; \tag{62}$$
$$\hat{s}(Y_t(g^{-1}) + Ad(g^{-1})A_+ - A_+)|_{\mathcal{H}^*} + ad^*(Y_t(g^{-1}) + Ad(g^{-1})A_+ - A_+)A_-|_{\mathcal{H}^*}$$
$$+ ad^*(Ad(h_R^{-1})E_-)(Ad^*(g^{-1}h_L)\mathcal{E}_+)|_{\mathcal{H}^*} = 0\,. \tag{63}$$

Projecting Eq. (61) along $\mathcal{H}^*$ and taking into account (62) we get that the expression in the second line of (61) (which lies entirely in the subspace $\mathcal{H}^*$) vanishes separately:

$$\partial_t A_- - \hat{s}(A_+) - ad^*(A_+)A_- = 0\,. \tag{64}$$

Equation (64) has the meaning of a generalized "zero-curvature" equation where we remind that $A_+ \in \mathcal{H}$ whereas $A_- \in \mathcal{H}^*$ (cf. (56)). Indeed, in the special case of Kac–Moody groups (31) it reduces to the ordinary zero-curvature equation on the subalgebra $\mathcal{H}$:

$$\partial_t A_- - \partial_x A_+ - [A_+, A_-] = 0\,, \quad A_- = \partial_x h_R^{-1} h_R\,, \quad A_+ = \partial_t h_L h_L^{-1}\,. \tag{65}$$

Now, going back to general coadjoint-orbit actions and taking into account (56) we see that Eq. (64) implies $h_L = h_R^{-1} = h$ where h is arbitrary element of the gauge subgroup H reflecting the residual gauge freedom. Therefore, we are entitled to choose the gauge fixing:

$$h = \mathbb{1} \to A_+ = 0\,, \qquad A_- = 0\,, \tag{66}$$

which simplifies the rest of the equations of motions to:

$$\partial_t S(g) + ad^*(Ad(g)E_-)\mathcal{E}_+ = 0\,; \tag{67}$$
$$\hat{s}(Y_t(g^{-1}))|_{\mathcal{H}^*} + ad^*(E_-)(Ad^*(g^{-1})\mathcal{E}_+)|_{\mathcal{H}^*} = 0\,. \tag{68}$$

Using relation (14) we can rewrite Eq. (67) in an equivalent form:

$$\hat{s}(Y_t(g^{-1})) - ad^*(E_-)(Ad^*(g^{-1})\mathcal{E}_+) = 0\,. \tag{69}$$

Comparing Eq. (68) and the projected along $\mathcal{H}^*$ Eq. (69), we deduce that both terms on the l.h.s. of (68) must separately vanish:

$$\hat{s}(Y_t(g^{-1})|_{\mathcal{H}^*} = 0\,, \qquad ad^*(E_-)(Ad^*(g^{-1})\mathcal{E}_+)|_{\mathcal{H}^*} = 0 \tag{70}$$

or, written in an equivalent form:

$$\begin{gathered} \langle \hat{s}(Y_t(g^{-1})|X_H\rangle = 0\,, \\ \langle ad^*(E_-)(Ad^*(g^{-1})\mathcal{E}_+)|X_H\rangle = -\langle Ad^*(g^{-1})\mathcal{E}_+|ad(E_-)X_H\rangle = 0\,. \end{gathered} \tag{71}$$

The consistency of the second condition in Eq. (71) is guaranteed because of the properties (50) of the fixed elements $\mathcal{E}_+ \in \mathcal{G}^*$ and $E_- \in \mathcal{H} \subset \mathcal{G}$. Note also that due to properties (50) the projection along $\mathcal{H}^*$ of the second term on the l.h.s. of Eq. (67) similarly vanishes: $ad^*(Ad(g)E_-)\mathcal{E}_+|_{\mathcal{H}^*} = 0$. Thus, taking into account

(70), the final form of the gauge-fixed equations of motion of the gauged geometric action (58) reads:

$$\partial_t S(g) + ad^*(Ad(g)E_-)\mathcal{E}_+ = 0\,, \quad \partial_t S(g)|_{\mathcal{H}^*} = 0\,, \quad \hat{s}(Y_t(g^{-1}))|_{\mathcal{H}^*} = 0\,. \tag{72}$$

5. "Zero-Curvature" Representation on Coadjoint Orbits

Let us now show that the first Eq. (72) together with the constraints (second and third Eqs. (72)) possess a generalized "zero-curvature" representation. To this end we introduce, in the spirit of Chapter 4 of Ref. 29, an additional loop-grading ("current-algebra") structure on the underlying (infinite-dimensional) algebra $\mathcal{G}$ (recall Sec. 3 for the various interesting examples of algebras $\mathcal{G}$) such that the original $\mathcal{G}$ is the zero-grade subalgebra:

$$\widehat{\mathcal{G}} = \oplus_{i\in\mathbb{Z}}\mathcal{G}^{(i)}\,, \quad \left[\mathcal{G}^{(i)}, \mathcal{G}^{(j)}\right] \subset \mathcal{G}^{(i+j)}\,, \quad \mathcal{G}^{(0)} = \mathcal{G}\,. \tag{73}$$

The loop-grading structure is induced on the corresponding dual space:

$$\widehat{\mathcal{G}}^* = \oplus_{i\in\mathbb{Z}}\mathcal{G}^*_{(i)}\,, \quad \left\langle \mathcal{G}^*_{(i)} \middle| \mathcal{G}^{(j)} \right\rangle = 0 \quad \text{for } i+j \neq 0\,. \tag{74}$$

The definitions of the basic objects — the central extension operator $\hat{s}(\cdot)$, the group cocycle $S(g)$ and the fundamental Maurer–Cartan form $Y(g)$ from Subsec. 2.1, naturally are carried over to the whole graded algebra $\widehat{\mathcal{G}}$ and the corresponding group $\widehat{G}$. Note that $\hat{s}(\cdot)$ preserves the grading structure.

Let us consider a group element $\hat{g} \in \widehat{G}$ of the following form:

$$\hat{g} = g\Omega\,, \quad g = e^{X^{(0)}} \in G\,, \quad \Omega = e^{\sum_{i\geq 1}\omega^{(i)}}\,, \tag{75}$$

where $\omega^{(j)} \in \mathcal{G}^{(j)}$, and let us introduce the following overdetermined system of equations for $\hat{g}$ given entirely in terms of the basic objects connected with a $\widehat{G}$-coadjoint orbit:

$$S(\hat{g}) - S(g) + \mathcal{E}_+^{(1)} = 0\,, \quad Y_t(\hat{g}^{-1}) + Ad(g\hat{g}^{-1})E_-^{(-1)} - E_-^{(-1)} = 0\,. \tag{76}$$

Here $E_-^{(-1)} \in \mathcal{G}^{(-1)}$ and $\mathcal{E}_+^{(1)} \in \mathcal{G}^*_{(1)}$ are fixed elements possessing the same properties as (50) but extended to the whole graded algebra:

$$\widehat{\mathcal{H}} \subset \widehat{\mathcal{G}}\,, \quad \widehat{\mathcal{H}} = \mathrm{Ker}\big(\mu(\mathcal{E}_+^{(1)})\big)\,, \quad E_-^{(-1)} \in \widehat{\mathcal{H}}\,, \quad \hat{s}\big(E_-^{(-1)}\big) = 0\,. \tag{77}$$

The element $\hat{g}$ (75) is a generalization of the notion of "monodromy matrix," and the system (76) can be viewed as generalization of the "zero-curvature" equations on coadjoint orbits of arbitrary (infinite-dimensional) groups with central extensions.

To solve the system (76) we insert the grade-expansion (75) and use the general cocycle properties (6) and (13). The lowest grade $= 1$ projection in the first Eq. (76) and the lowest (nontrivial) grade $= 0$ projection in the second Eq. (76) yield the following overdetermined *linear* system of equations for the lowest positive-grade Lie-algebraic component $\omega^{(1)}$ in $\hat{g}$ (75):

$$\hat{s}\big(\omega^{(1)}\big) + Ad^*\big(g^{-1}\big)\mathcal{E}_+^{(1)} = 0\,, \qquad Y_t\big(g^{-1}\big) - ad\big(\omega^{(1)}\big)E_-^{(-1)} = 0\,. \tag{78}$$

Now, acting with the operator $\hat{s}(\cdot)$ on the second Eq. (78) and using the first Eq. (78) together with (3) and last Eq. (77) we obtain

$$\hat{s}\big(Y_t\big(g^{-1}\big)\big) - ad^*\big(E_-^{(-1)}\big)\big(Ad^*(g^{-1})\mathcal{E}_+^{(1)}\big) = 0\,, \tag{79}$$

which upon using (14) can be rewritten in the equivalent form:

$$\partial_t S(g) + ad^*(Ad(g)E_-^{(-1)})\mathcal{E}_+^{(1)} = 0\,. \tag{80}$$

The latter equation coincides with first Eq. (72) upon identifying $\mathcal{E}_+^{(1)} = \mathcal{E}_+$ and $E_-^{(-1)} = E_-$. The constraints (second and third Eqs. (72)) are recovered from (79)–(80) by projecting along $\widehat{\mathcal{H}}$ and using properties (77).

In the special case of Kac–Moody group G (see example 3.1), Eqs. (76) acquire the form:

$$\big(\partial_x + E_+^{(1)} + A^{(0)}\big)\hat{g} = 0\,, \qquad A^{(0)} \equiv -\partial_x g g^{-1}\,, \tag{81}$$

$$\bar{\partial}\hat{g} + \hat{g}E_-^{(-1)} - \big(gE_-^{(-1)}g^{-1}\big)\hat{g} = 0\,, \qquad \bar{\partial} \equiv \partial_t\,, \tag{82}$$

where $E_\pm^{(\pm1)} \in \widehat{\mathcal{H}} = \mathrm{Ker}(ad(E_+^{(1)}))$. The last equation can be written, using the grade structure (75) of $\hat{g}$, entirely in terms of the latter:

$$(\bar{\partial} + (\hat{g}E_-^{(-1)}\hat{g}^{-1})_+)\hat{g} = 0\,, \tag{83}$$

with the subscript $(+)$ indicating projection along the nonnegative grade part. The zero-curvature (compatibility) condition for the system (81), (83):

$$[\bar{\partial} + (\hat{g}E_-^{(-1)}\hat{g}^{-1})_+,\ \partial_x + E_+^{(1)} + A^{(0)}] = 0 \tag{84}$$

upon inserting the grade-expansion of $\hat{g}$ (75) reduces to the form:

$$[\bar{\partial} - gE_-^{(-1)}g^{-1},\ \partial_x + E_+^{(1)} + A^{(0)}] = 0\,, \tag{85}$$

which yields:

$$\bar{\partial}(\partial_x g g^{-1}) - [E_+^{(1)}, gE_-^{(-1)}g^{-1}] = 0\,, \tag{86}$$

$$\bar{\partial}(\partial_x g g^{-1})|_{\mathcal{H}} = 0 \to \partial_x g g^{-1}|_{\mathcal{H}} = -A^{(0)}|_{\mathcal{H}} = \xi_H(x)\,, \tag{87}$$

$$\partial(g^{-1}\bar{\partial}g)|_{\mathcal{H}} = 0 \to g^{-1}\bar{\partial}g|_{\mathcal{H}} = ([E_-^{(-1)}, \omega^{(1)}]|_{\mathcal{H}}) = \eta_H(t)\,, \tag{88}$$

where $\xi_H(x)$ and $\eta_H(t)$ are some fixed nondynamical elements of $\mathcal{H} \subset \mathcal{G}$. The equality in brackets in (88) is a special case of second Eq. (78).

In the context of integrable systems, when $E_+^{(1)}$ is a semisimple element (55), the constraints (87)–(88) can be brought to a simpler form with vanishing $\xi_H(x)$ and $\eta_H(t)$ (cf. Eqs. (189)–(190) below).

Equation (85) constitutes the well-known zero-curvature representation for the equations of motion (86) of non-Abelian Toda field theories[16] and more general G/H gauged WZNW models[30] with various special choices for the fixed algebraic elements $E_+^{(1)}$ and $E_-^{(-1)}$; for a systematic treatment of non-Abelian Toda field theories as gauge-fixed versions of gauged WZNW models, see Ref. 31.

There is also another way to view Eqs. (86), namely, we will see below that (86) arise as *additional symmetry* flow equations of generalized Drinfeld–Sokolov integrable hierarchies. In the special case of generalized Drinfeld–Sokolov hierarchies based on $\widehat{\mathcal{G}} = \widehat{\mathrm{SL}}(M+1)$ with standard homogeneous grading $Q = \lambda\frac{\partial}{\partial\lambda}$ and $E_{\pm}^{(\pm 1)} = H_{\lambda_M}^{(\pm 1)}$ (where $H_{\lambda_M} \equiv \lambda_M.H$ with λ_M being the last fundamental weight), which are equivalent to the class of constrained (reduced) KP hierarchies $c\mathrm{KP}_{1,M}$ (Eq. (192) below) within the Sato pseudo-differential operator formulation, we will be able to use the standard Darboux–Bäcklund techniques to generate solutions to gauged $\mathrm{SL}(M+1)/\mathrm{U}(1) \times \mathrm{SL}(M)$ WZNW field equation (86).

6. Sato Formalism for Additional Symmetries of KP-Type Integrable Hierarchies

6.1. *Sato pseudo-differential operator formulation*

In what follows D denotes the derivative operator w.r.t. x such that $[D, f] = \partial f = \partial f/\partial x$ and the generalized Leibniz rule holds: $D^n f = \sum_{j=0}^{\infty} \binom{n}{j} (\partial^j f) D^{n-j}$ with $n \in \mathbb{Z}$. In order to avoid confusion we shall employ the following notations: for any (pseudo-)differential operator $A = \sum_k a_k D^k$ and a function f, the symbol $A(f)$ will indicate application (action) of A on f, whereas the symbol Af will denote simply operator product of A with the zero-order (multiplication) operator f. Projections $(\pm)$ are defined as: $A_+ = \sum_{k\geq 0} a_k D^k$ and $A_- = \sum_{k\leq -1} a_k D^k$. Conjugation is given by $A^* = \sum_k (-D)^k a_k$. Finally, $\mathrm{Res}\, A \equiv a_{-1}$.

The general one-component (scalar) KP hierarchy is given by a pseudo-differential Lax operator $\mathcal{L}$ obeying Sato evolution equations (also known as isospectral flow equations; for a systematic exposition, see Ref. 18):

$$\mathcal{L} = D + \sum_{k=1}^{\infty} u_k D^{-k}\,, \qquad \frac{\partial}{\partial t_n}\mathcal{L} = [(\mathcal{L}^n)_+,\, \mathcal{L}]\,, \tag{89}$$

with Sato dressing operator W:

$$\mathcal{L} = WDW^{-1}\,, \quad \frac{\partial}{\partial t_n} W = -(WD^n W^{-1})_- W\,, \quad W = \sum_{k=0}^{\infty} \frac{p_k(-[\partial])\tau(t)}{\tau(t)} D^{-k}\,, \tag{90}$$

and (adjoint) Baker–Akhiezer (BA) wave functions $\psi_{\mathrm{BA}}^{(*)}(t,\lambda)$:

$$\mathcal{L}^{(*)}\psi_{\mathrm{BA}}^{(*)} = \lambda\psi_{\mathrm{BA}}^{(*)}\,, \quad \frac{\partial}{\partial t_n}\psi_{\mathrm{BA}}^{(*)} = \pm({\mathcal{L}^{(*)}}^n)_+(\psi_{\mathrm{BA}}^{(*)})\,, \tag{91}$$

$$\psi_{\mathrm{BA}}^{(*)}(t,\lambda) = W^{(*-1)}(e^{\pm\xi(t,\lambda)}) = \frac{\tau(t \mp [\lambda^{-1}])}{\tau(t)} e^{\pm\xi(t,\lambda)}\,, \quad \xi(t,\lambda) \equiv \sum_{\ell=1}^{\infty} t_\ell \lambda^\ell\,. \tag{92}$$

Here and below we employ the following shorthand notations: $(t) \equiv (t_1 \equiv x, t_2, \ldots)$ for the set of isospectral time-evolution parameters; $[\partial] \equiv \left(\frac{\partial}{\partial t_1}, \frac{1}{2}\frac{\partial}{\partial t_2}, \frac{1}{3}\frac{\partial}{\partial t_3}, \ldots\right)$ and $[\lambda^{-1}] \equiv \left(\lambda^{-1}, \frac{1}{2}\lambda^{-2}, \frac{1}{3}\lambda^{-3}, \ldots\right)$; $p_k(\cdot)$ indicate the well-known Schur polynomials.

The tau-function $\tau(t)$ is related to the coefficients of the Lax operator (89) through the relation:

$$\partial_x \frac{\partial}{\partial t_n} \ln \tau = \mathrm{Res}\, \mathcal{L}^n \,, \tag{93}$$

where the terms on the r.h.s. of (93) are the densities of the conserved quantities.

There exist a few other objects in Sato formalism for integrable hierarchies which play a fundamental role in our construction. (Adjoint) eigenfunctions $\Phi(t)$ ($\Psi(t)$, respectively) are those functions of KP "times" (t) satisfying:

$$\frac{\partial}{\partial t_l}\Phi = (\mathcal{L}^l)_+(\Phi)\,, \qquad \frac{\partial}{\partial t_l}\Psi = -(\mathcal{L}^l)^*_+(\Psi)\,. \tag{94}$$

According to second equation (91), (adjoint) BA functions are special cases of (adjoint) eigenfunctions, which in addition satisfy spectral equations (first equation (91)).

It has been shown in Ref. 32 that any (adjoint) eigenfunction possesses a spectral representation of the form:[b]

$$\Phi(t) = \int d\lambda\, \varphi(\lambda)\psi_{\mathrm{BA}}(t,\lambda)\,, \qquad \Psi(t) = \int d\lambda\, \psi(\lambda)\psi^*_{\mathrm{BA}}(t,\lambda)\,, \tag{95}$$

with appropriate spectral densities $\varphi(\lambda)$ and $\psi(\lambda)$ which are formal Laurent series in λ. Clearly, any KP hierarchy possesses an infinite set of independent (adjoint) eigenfunctions in one-to-one correspondence with the space of all independent formal Laurent series in λ.

The next important object is the so-called squared eigenfunction potential (SEP)[33] — a function $S(\Phi(t),\Psi(t))$ associated with an arbitrary pair of (adjoint) eigenfunctions $\Phi(t),\Psi(t)$ which possesses the following characteristics:

$$\frac{\partial}{\partial t_n} S(\Phi(t),\Psi(t)) = \mathrm{Res}(D^{-1}\Psi(\mathcal{L}^n)_+\Phi D^{-1})\,. \tag{96}$$

In particular, for $n=1$ Eq. (96) implies $\partial_x S(\Phi(t),\Psi(t)) = \Phi(t)\Psi(t)$ (recall $\partial_x \equiv \partial/\partial t_1$). Equation (96) determines $S(\Phi(t),\Psi(t)) \equiv \partial^{-1}(\Phi(t)\Psi(t))$ up to a shift by a trivial constant which is uniquely fixed by the fact that any SEP obeys the following double-spectral representation:[32]

$$\begin{aligned}\partial^{-1}(\Phi(t)\Psi(t)) &= -\iint d\lambda\, d\mu\, \psi(\lambda)\varphi(\mu)\frac{1}{\lambda}\psi^*_{\mathrm{BA}}(t,\lambda)\psi_{\mathrm{BA}}(t+[\lambda^{-1}],\mu) \\ &= -\iint d\lambda\, d\mu\, \frac{\psi(\lambda)\varphi(\mu)}{\lambda-\mu} e^{\xi(t,\mu)-\xi(t,\lambda)}\frac{\tau(t+[\lambda^{-1}]-[\mu^{-1}])}{\tau(t)}\,,\end{aligned} \tag{97}$$

with $\varphi(\lambda)$, $\psi(\lambda)$ being the respective spectral densities in (95). It is in this well-defined sense that inverse space derivatives ∂^{-1} will appear throughout our construction below.

[b]Integrals over spectral parameters are understood as: $\int d\lambda \equiv \oint_0 \frac{d\lambda}{2i\pi} = \mathrm{Res}_{\lambda=0}$.

For later use we recall some further relations obeyed by SEP functions involving (adjoint) BA functions:[32]

$$
\begin{aligned}
\partial^{-1}(\psi_{\mathrm{BA}}(t,\lambda)\Psi(t)) &= \frac{1}{\lambda}\psi_{\mathrm{BA}}(t,\lambda)\Psi(t-[\lambda^{-1}])\,,\\
\partial^{-1}(\psi^{*}_{\mathrm{BA}}(t,\lambda)\Phi(t)) &= -\frac{1}{\lambda}\psi^{*}_{\mathrm{BA}}(t,\lambda)\Phi(t+[\lambda^{-1}])\,,
\end{aligned}
\tag{98}
$$

where $\Phi(t)$ and $\Psi(t)$ are arbitrary (adjoint) eigenfunctions (94).

In what follows we shall make an essential use of the well-known pseudo-differential operator identities (cf. e.g. the appendix in first Ref. 34):

$$
\begin{gathered}
(\mathcal{B}M)_{-} = \mathcal{B}(f)D^{-1}g\,, \qquad (M\mathcal{B})_{-} = fD^{-1}\mathcal{B}^{*}(g)\,,\\
M_1M_2 = M_1(f_2)D^{-1}g_2 + f_1D^{-1}M_2^{*}(g_1)\,, \qquad M \equiv fD^{-1}g\,,\\
M_{1,2} \equiv f_{1,2}D^{-1}g_{1,2}\,, \qquad M_1(f_2) = f_1\partial^{-1}(g_1f_2)\ \text{etc.}\,,
\end{gathered}
\tag{99}
$$

where $\mathcal{B}$ is arbitrary purely differential operator.

6.2. *Constrained* KP *hierarchies. Inverse powers of Lax operators*

So far we have considered the general case of unconstrained KP hierarchy. Now we are interested in symmetries for *constrained* KP hierarchies $c\mathrm{KP}_{R,M}$ with Lax operators (cf. Refs. 34, 32 and 35 and references therein):[c]

$$
\mathcal{L} \equiv \mathcal{L}_{R,M} = D^R + \sum_{i=0}^{R-2} u_iD^i + \sum_{j=1}^{M}\Phi_jD^{-1}\Psi_j = L_{M+R}L_M^{-1}\,,
\tag{100}
$$

where $\{\Phi_i,\Psi_i\}_{i=1}^{M}$ is a set of (adjoint) eigenfunctions of $\mathcal{L}$. The class of $c\mathrm{KP}_{R,M}$ hierarchies (100) contains various well-known integrable hierarchies as special cases: mKdV hierarchies (for $M=0$); AKNS hierarchy (for $R=1, M=1$); Fordy–Kulish hierarchies[39] (for $R=1$, M arbitrary); Yajima–Oikawa equations (for $R=2$, $M=1$), etc.

The second representation of $\mathcal{L} \equiv \mathcal{L}_{R,M}$[d] is in terms of a ratio of two monic purely differential operators L_{M+R} and L_M of orders $M+R$ and M, respectively (see Ref. 35 and references therein). For $\mathcal{L} \equiv \mathcal{L}_{R,M}$ the Sato evolution (isospectral

[c]Originally $c\mathrm{KP}_{R,M}$ hierarchies appeared in different disguises from various parallel developments: (i) symmetry reductions of the general unconstrained KP hierarchy;[36,33] (ii) free-field realizations, in terms of finite number of fields, of both compatible first and second KP Hamiltonian structures;[37] (iii) a method of extracting continuum integrable hierarchies from generalized Toda-like lattice hierarchies underlying (multi-)matrix models in string theory.[38]

[d]Henceforth we shall employ the short-hand notation $\mathcal{L}$ for $\mathcal{L}_{R,M}$ (100) whenever this will not lead to a confusion.

flow) Eqs. (89), the equations for (adjoint) BA (91) and (adjoint) eigenfunctions (94) acquire the form:

$$\frac{\partial}{\partial t_n}\mathcal{L} = \left[(\mathcal{L}^{\frac{n}{R}})_+, \mathcal{L}\right] , \quad \mathcal{L}^{(*)}\psi^{(*)}_{\mathrm{BA}} = \lambda^R \psi^{(*)}_{\mathrm{BA}} ,$$
$$\frac{\partial}{\partial t_n}\psi^{(*)}_{\mathrm{BA}} = \pm(\mathcal{L}^{(*)})_+^{\frac{n}{R}}(\psi^{(*)}_{\mathrm{BA}}) , \tag{101}$$

$$\frac{\partial}{\partial t_n}\Phi = (\mathcal{L}^{\frac{n}{R}})_+(\Phi) , \quad \frac{\partial}{\partial t_n}\Psi = -(\mathcal{L}^{\frac{n}{R}})^*_+(\Psi) . \tag{102}$$

In what follows we will also need the explicit form of inverse powers of the Lax operator $\mathcal{L} = L_{M+R}L_M^{-1}$ (100). First, let us recall that the inverses of the underlying purely differential operators are given by

$$L_M^{-1} = \sum_{i=1}^{M} \varphi_i D^{-1}\psi_i , \qquad L_{M+R}^{-1} = \sum_{a=1}^{M+R} \bar{\varphi}_a D^{-1}\bar{\psi}_a , \tag{103}$$

where the functions $\{\varphi_i\}_{i=1}^M$ and $\{\psi_i\}_{i=1}^M$ span $\mathrm{Ker}(L_M)$ and $\mathrm{Ker}(L_M^*)$, respectively, whereas $\{\bar{\varphi}_a\}_{a=1}^{M+R}$ and $\{\bar{\psi}_a\}_{a=1}^{M+R}$ span $\mathrm{Ker}(L_{M+R})$ and $\mathrm{Ker}(L^*_{M+R})$, respectively. Therefore we have

$$\mathcal{L} = \mathcal{L}_+ + \sum_{i=1}^{M} L_{M+R}(\varphi_i)D^{-1}\psi_i , \quad \text{i.e. } \Phi_i = L_{M+R}(\varphi_i) , \quad \Psi_i = \psi_i , \tag{104}$$

$$\mathcal{L}^{-1} = \sum_{a=1}^{M+R} L_M(\bar{\varphi}_a)D^{-1}\bar{\psi}_a , \tag{105}$$

$$\mathcal{L}^{-N} = \sum_{a=1}^{M+R}\sum_{s=0}^{N-1} \mathcal{L}^{-(N-1)+s}(L_M(\bar{\varphi}_a))D^{-1}(\mathcal{L}^{-s})^*(\bar{\psi}_a) . \tag{106}$$

The last formula (106) is completely analogous in structure with the formula[40] for the negative pseudo-differential part of a positive power of $\mathcal{L}$ (100):

$$(\mathcal{L}^N)_- = \sum_{i=1}^{M}\sum_{s=0}^{N-1} \mathcal{L}^{N-1-s}(\Phi_i)D^{-1}(\mathcal{L}^s)^*(\Psi_i) . \tag{107}$$

Let us also note that the following simple consequences from the definitions of the corresponding objects will play essential role for the consistency of the constructions involving inverse powers of $\mathcal{L}$:

$$\mathcal{L}(L_M(\bar{\varphi}_a)) = 0 , \quad \mathcal{L}^*(\bar{\psi}_a) = 0 , \quad \mathcal{L}^{-1}(\Phi_i) = 0 , \quad (\mathcal{L}^{-1})^*(\Psi_i) = 0 . \tag{108}$$

Applying the isospectral flow Eqs. (89) to $\mathcal{L}^{-1}$, i.e. $\partial/\partial t_n \mathcal{L}^{-1} = [(\mathcal{L}^{\frac{n}{R}})_+, \mathcal{L}^{-1}]$ and taking into account the explicit form of $\mathcal{L}^{-1}$ (105) we find that $L_M(\bar{\varphi}_a)$ and $\bar{\psi}_a$ are (adjoint) eigenfunctions of $\mathcal{L}$ (cf. (102)):

$$\frac{\partial}{\partial t_n}L_M(\bar{\varphi}_a) = (\mathcal{L}^{\frac{n}{R}})_+(L_M(\bar{\varphi}_a)) , \qquad \frac{\partial}{\partial t_n}\bar{\psi}_a = -(\mathcal{L}^{\frac{n}{R}})^*_+(\bar{\psi}_a) . \tag{109}$$

6.3. *Loop-algebra symmetries of* KP *hierarchies*

Let us consider the following system of M infinite sets of (adjoint) eigenfunctions of $\mathcal{L} \equiv \mathcal{L}_{R,M}$ (100):

$$\Phi_i^{(n)} \equiv \mathcal{L}^{n-1}(\Phi_i)\,, \quad \Psi_i^{(n)} \equiv (\mathcal{L}^*)^{n-1}(\Psi_i)\,, \qquad n = 1, 2, \ldots; \quad i = 1, \ldots, M\,, \tag{110}$$

which are expressed in terms of the M pairs of (adjoint) eigenfunctions entering the pseudo-differential part of $\mathcal{L} \equiv \mathcal{L}_{R,M}$ (100). Using (110) we can build the following infinite set of additional symmetry flows:

$$\delta_A^{(n)} \mathcal{L} = [\mathcal{M}_A^{(n)}, \mathcal{L}]\,, \quad \mathcal{M}_A^{(n)} \equiv \sum_{i,j=1}^{M} A_{ij}^{(n)} \sum_{s=1}^{n} \Phi_j^{(n+1-s)} D^{-1} \Psi_i^{(s)}\,, \tag{111}$$

where $A^{(n)}$ is an arbitrary constant $M \times M$ matrix, i.e. $A^{(n)} \in \mathrm{Mat}(M)$. The flows (111) define symmetries since they commute with the isospectral flows $\frac{\partial}{\partial t_l}$:

$$\left[\delta_\alpha, \frac{\partial}{\partial t_l}\right] = 0 \leftrightarrow \frac{\partial}{\partial t_l} \mathcal{M}_\alpha = [(\mathcal{L}^l)_+, \mathcal{M}_A^{(n)}]_-\,. \tag{112}$$

The last evolution equation in (112) is directly verified upon using pseudo-differential identities (99) and the fact all the functions entering the definition of $\mathcal{M}_A^{(n)}$ (111) are (adjoint) eigenfunctions obeying (94).

Consistency of the flow action (111) with the constrained form (100) of $\mathcal{L} \equiv \mathcal{L}_{R,M}$ implies the following flow action on the involved (adjoint) eigenfunctions:

$$\begin{aligned} \delta_A^{(n)} \Phi_i^{(m)} &= \mathcal{M}_A^{(n)}(\Phi_i^{(m)}) - \sum_{j=1}^{M} A_{ij}^{(n)} \Phi_j^{(n+m)}\,, \\ \delta_A^{(n)} \Psi_i^{(m)} &= -(\mathcal{M}_A^{(n)})^*(\Psi_i^{(m)}) + \sum_{j=1}^{M} A_{ji}^{(n)} \Psi_j^{(n+m)}\,. \end{aligned} \tag{113}$$

The specific form of the inhomogeneous terms on the r.h.s. of Eqs. (113) is the main ingredient of our symmetry flow construction. It is precisely these inhomogeneous terms which yield nontrivial loop-algebra additional symmetries.

Using the pseudo-differential operator identities (99) and taking into account (113) we can show that:

$$\delta_A^{(n)} \mathcal{M}_B^{(m)} - \delta_B^{(m)} \mathcal{M}_A^{(n)} - [\mathcal{M}_A^{(n)}, \mathcal{M}_B^{(m)}] = \mathcal{M}_{[A,B]}^{(n+m)}\,. \tag{114}$$

Equation (114) implies that the symmetry flows (111)–(113) span the following infinite-dimensional algebra:

$$[\delta_A^{(n)}, \delta_B^{(m)}] = \delta_{[A,B]}^{(n+m)}\,; \quad A^{(n)},\ B^{(m)} \in \mathrm{Mat}(M)\,, \quad n, m = 1, 2, \ldots \tag{115}$$

isomorphic to $(\widehat{U}(1) \times \widehat{\mathrm{SL}}(M))_+$ where the subscript $(+)$ indicates taking the positive-grade subalgebra of the corresponding loop-algebra. We observe, that in the case of $c\mathrm{KP}_{R,M}$ models we have $\mathcal{M}_{A=\mathbb{1}}^{(n)} = (\mathcal{L}_{R,M}^n)_-$ (insert (110) into first relation (111) for $A^{(n)} = \mathbb{1}$ and compare with (107)). Therefore, the flows $\delta_{A=\mathbb{1}}^{(n)}$ for

$c\mathrm{KP}_{R,M}$ models coincide upto a sign with the ordinary isospectral flows modulo R: $\delta^{(n)}_{A=1\!\!1} = -\frac{\partial}{\partial t_{nR}}$. Thereby the flows $\delta^{(n)}_A$ (111) will be called "positive" for brevity.

Now we consider another infinite set of (adjoint) eigenfunctions of $\mathcal{L} \equiv \mathcal{L}_{R,M}$ expressed in terms of the (adjoint) eigenfunctions entering the inverse power of $\mathcal{L}^{-1} \equiv \mathcal{L}^{-1}_{R,M}$ (105):

$$\begin{gathered}\Phi^{(-m)}_a \equiv \mathcal{L}^{-(m-1)}(L_M(\bar{\varphi}_a))\,, \qquad \Psi^{(-m)}_a \equiv (\mathcal{L}^{-(m-1)})^*(\bar{\psi}_a)\,, \\ m = 1, 2, \dots\,, \qquad a = 1, \dots, M+R\,. \end{gathered} \tag{116}$$

Using (116) we obtain the following set of "negative" symmetry flows which parallels completely the set of "positive" flows (111):

$$\delta^{(-n)}_{\mathcal{A}}\mathcal{L} = [\mathcal{M}^{(-n)}_{\mathcal{A}}, \mathcal{L}]\,, \quad \mathcal{M}^{(-n)}_{\mathcal{A}} \equiv \sum_{a,b=1}^{M+R} \mathcal{A}^{(-n)}_{ab} \sum_{s=1}^{n} \Phi^{(-n-1+s)}_b D^{-1} \Psi^{(-s)}_a\,, \tag{117}$$

where $\mathcal{A}^{(-n)}_{ab}$ is an arbitrary constant $(M+R) \times (M+R)$ matrix, i.e. $\mathcal{A}^{(-n)} \in \mathrm{Mat}(M+R)$. In fact, since according to (106) we have $\mathcal{M}^{(-n)}_{\mathcal{A}=1\!\!1} = \mathcal{L}^{-n}$, the flows $\delta^{(-n)}_{\mathcal{A}=1\!\!1}$ vanish identically, i.e. $\delta^{(-n)}_{\mathcal{A}=1\!\!1} = 0$, therefore, we restrict $\mathcal{A}^{(-n)} \in \mathrm{SL}(M+R)$.

Consistency of the flow action (117) with the constrained form (100) of $\mathcal{L} \equiv \mathcal{L}_{R,M}$ and with the constrained form (105) of the inverse $\mathcal{L}^{-1}$ implies the following $\delta^{(-n)}_{\mathcal{A}}$-flow action on the involved (adjoint) eigenfunctions (using shorthand notations (110) and (116)):

$$\delta^{(-n)}_{\mathcal{A}}\Phi^{(m)}_i = \mathcal{M}^{(-n)}_{\mathcal{A}}(\Phi^{(m)}_i)\,, \qquad \delta^{(-n)}_{\mathcal{A}}\Psi^{(m)}_i = -(\mathcal{M}^{(-n)}_{\mathcal{A}})^*(\Psi^{(m)}_i)\,, \tag{118}$$

$$\begin{aligned}\delta^{(-n)}_{\mathcal{A}}\Phi^{(-m)}_a &= \mathcal{M}^{(-n)}_{\mathcal{A}}(\Phi^{(-m)}_a) - \sum_{b=1}^{M+R} \mathcal{A}^{(-n)}_{ab}\Phi^{(-n-m)}_b\,, \\ \delta^{(-n)}_{\mathcal{A}}\Psi^{(-m)}_a &= -(\mathcal{M}^{(-n)}_{\mathcal{A}})^*(\Psi^{(-m)}_a) + \sum_{b=1}^{M+R} \mathcal{A}^{(-n)}_{ba}\Psi^{(-n-m)}_b\,. \end{aligned} \tag{119}$$

Similarly, consistency of "positive" $\delta^{(n)}_A$-flow action (111) with the constrained form (105) of the inverse Lax operator implies:

$$\delta^{(n)}_A\Phi^{(-m)}_a = \mathcal{M}^{(n)}_A(\Phi^{(-m)}_a)\,, \qquad \delta^{(n)}_A\Psi^{(-m)}_a = -(\mathcal{M}^{(n)}_A)^*(\Psi^{(-m)}_a)\,. \tag{120}$$

Using again the pseudo-differential operator identities (99) we find from (118)–(120) (cf. Eq. (114)):

$$\delta^{(n)}_A\mathcal{M}^{(-m)}_{\mathcal{B}} - \delta^{(-m)}_{\mathcal{B}}\mathcal{M}^{(n)}_A - [\mathcal{M}^{(n)}_A, \mathcal{M}^{(-m)}_{\mathcal{B}}] = 0\,, \tag{121}$$

$$\delta^{(-n)}_{\mathcal{A}}\mathcal{M}^{(-m)}_{\mathcal{B}} - \delta^{(-m)}_{\mathcal{B}}\mathcal{M}^{(-n)}_{\mathcal{A}} - [\mathcal{M}^{(-n)}_{\mathcal{A}}, \mathcal{M}^{(-m)}_{\mathcal{B}}] = \mathcal{M}^{(-n-m)}_{[\mathcal{A},\mathcal{B}]}\,. \tag{122}$$

Equations (121)–(122) imply that the "negative" symmetry flows (117)–(119) commute with the "positive" flows (111)–(113):

$$[\delta^{(n)}_A, \delta^{(-m)}_{\mathcal{B}}] = 0\,, \tag{123}$$

and that they themselves span the following infinite-dimensional algebra:

$$[\delta^{(-n)}_{\mathcal{A}}, \delta^{(-m)}_{\mathcal{B}}] = \delta^{(-n-m)}_{[\mathcal{A},\mathcal{B}]}\,; \quad \mathcal{A}^{(-n)}, \mathcal{B}^{(-m)} \in \mathrm{SL}(M+R)\,, \quad n,m = 1,2,\ldots\,, \tag{124}$$

which is isomorphic to $(\widehat{\mathrm{SL}}(M+R))_-$ (the subscript $(-)$ indicates taking the negative-grade subalgebra of the corresponding loop-algebra).

Therefore, we conclude that the full loop algebra of (additional) symmetries of $c\mathrm{KP}_{R,M}$ hierarchies (100) is the direct sum:

$$(\widehat{\mathrm{U}}(1) \oplus \widehat{\mathrm{SL}}(M))_+ \oplus (\widehat{\mathrm{SL}}(M+R))_-\,, \tag{125}$$

where also the ordinary isospectral-flow symmetries are included.

Furthermore, starting from relation (93) and using (111) and (117) we find for the transformation laws under the $\delta^{(n)}_A$- and $\delta^{(-n)}_{\mathcal{A}}$-flow actions of the tau-function:

$$\delta^{(n)}_A \ln\tau = -\partial^{-1}(\mathrm{Res}\,\mathcal{M}^{(n)}_A) = -\sum_{i,j=1}^{M} A^{(n)}_{ij} \sum_{s=1}^{n} \partial^{-1}(\Phi^{(n+1-s)}_j \Psi^{(s)}_i)\,, \tag{126}$$

$$\delta^{(-n)}_{\mathcal{A}} \ln\tau = -\partial^{-1}(\mathrm{Res}\,\mathcal{M}^{(-n)}_{\mathcal{A}}) = -\sum_{a,b=1}^{M+R} \mathcal{A}^{(-n)}_{ab} \sum_{s=1}^{n} \partial^{-1}(\Phi^{(-n-1+s)}_b \Psi^{(-s)}_a)\,. \tag{127}$$

Using the double spectral representation (97) for SEP functions we can rewrite Eqs. (126)–(127) in the following equivalent form, namely, the action of additional symmetry flows on the tau-function is given by the action of the "smeared" Kyoto school bilocal vertex operator $\widehat{\mathcal{X}}(\lambda,\mu)$:[41]

$$\delta^{(n)}_A \tau(t) = \iint d\lambda\, d\mu\, \rho^{(n)}_A(\lambda,\mu) \widehat{\mathcal{X}}(\lambda,\mu)\tau(t)\,, \tag{128}$$

$$\delta^{(-n)}_{\mathcal{A}} \tau(t) = \iint d\lambda\, d\mu\, \rho^{(-n)}_{\mathcal{A}}(\lambda,\mu) \widehat{\mathcal{X}}(\lambda,\mu)\tau(t)\,. \tag{129}$$

Here

$$\rho^{(n)}_A(\lambda,\mu) \equiv \frac{\lambda^n - \mu^n}{\lambda-\mu} \sum_{i,j=1}^{M} A^{(n)}_{ij} \psi_i(\lambda)\varphi_j(\mu)\,, \tag{130}$$

$$\rho^{(-n)}_{\mathcal{A}}(\lambda,\mu) \equiv \frac{\lambda^{-n} - \mu^{-n}}{\lambda^{-1}-\mu^{-1}} \sum_{a,b=1}^{M+R} \mathcal{A}^{(-n)}_{ab} \psi^{(-1)}_a(\lambda)\varphi^{(-1)}_b(\mu)\,, \tag{131}$$

with $\varphi_i(\lambda)$, $\psi_i(\lambda)$ and $\varphi^{(-1)}_a(\lambda)$, $\psi^{(-1)}_a(\lambda)$ indicating the spectral densities (cf. (95)) of the (adjoint) eigenfunctions Φ_i, Ψ_i and $\Phi^{(-1)}_a \equiv L_M(\bar{\varphi}_a)$, $\Psi^{(-1)}_a \equiv \bar{\psi}_a$, respectively. Further, $\widehat{\mathcal{X}}(\lambda,\mu)$ denotes the Kyoto school bilocal vertex operator:[41]

$$\widehat{\mathcal{X}}(\lambda,\mu) = \frac{1}{\lambda-\mu} :e^{\hat{\theta}(\lambda)-\hat{\theta}(\mu)}: = \frac{1}{\lambda-\mu} e^{\xi(t,\mu)-\xi(t,\lambda)} e^{\sum_1^\infty \frac{1}{l}(\lambda^{-l}-\mu^{-l})\frac{\partial}{\partial t_l}} \tag{132}$$

for $|\mu| \le |\lambda|$, where

$$\hat{\theta}(\lambda) \equiv -\sum_{l=1}^{\infty} \lambda^l t_l + \sum_{l=1}^{\infty} \frac{1}{l}\lambda^{-l}\frac{\partial}{\partial t_l}\,, \tag{133}$$

and the columns $:\cdots:$ in (132) indicate Wick normal ordering w.r.t. the creation/annihilation "modes" t_l and $\frac{\partial}{\partial t_l}$, respectively.

Since the tau-function contains all solutions of the underlying integrable hierarchy, Eqs. (126)–(127) or Eqs. (128)–(129) describe the action of loop-algebra (125) additional symmetries on the space of (soliton-like) solutions of $c\mathrm{KP}_{R,M}$ hierarchies (100).

Remark. The construction above can be straightforwardly extended to the case of the general unconstrained KP hierarchy defined by (89). All relations (111)–(115) and (117)–(124) remain intact where now:

$$\left\{\Phi_i^{(n)}, \Psi_i^{(n)}\right\}_{i=1,\dots,M}^{n=1,2,\dots}, \quad \left\{\Phi_a^{(-n)}, \Psi_a^{(-n)}\right\}_{a=1,\dots,M+R}^{n=1,2,\dots} \tag{134}$$

form an infinite system of independent (adjoint) eigenfunctions of the general Lax operator (89) with M, $M+R$ being arbitrary positive integers.

6.4. *Multicomponent* KP *hierarchies from one-component ones*

Let us now consider the following subset of "positive" flows $\delta_{E_k}^{(n)}$ (111) for the general KP hierarchy (89) corresponding to:

$$E_k = \mathrm{diag}(0,\dots 0,1,0,\dots,0)\,, \qquad \text{i.e.} \quad \mathcal{M}_{E_k}^{(n)} = \sum_{s=1}^{n} \Phi_k^{(n+1-s)} D^{-1} \Psi_k^{(s)}\,. \tag{135}$$

Due to Eq. (115) the flows $\delta_{E_k}^{(n)}$ span an infinite-dimensional Abelian algebra and, by construction, they commute with the original isospectral flows $\frac{\partial}{\partial t_n}$ as well. Using the extended set of mutually commuting flows:

$$\frac{\partial}{\partial t_n} \equiv \partial/\partial \overset{(1)}{t}_n\,, \quad \delta_{E_k}^{(n)} \equiv \partial/\partial \overset{(k+1)}{t}_n\,, \quad k=1,\dots,M\,, \tag{136}$$

we can construct the following extended KP integrable hierarchy starting from (89):

$$\partial/\partial t_n \mathcal{L} = [(\mathcal{L}^n)_+, \mathcal{L}]\,, \qquad \partial/\partial \overset{(k)}{t}_n \mathcal{L} = [\mathcal{M}_{E_k}^{(n)}, \mathcal{L}] \tag{137}$$

with $\mathcal{M}_{E_k}^{(n)}$ as in (135), where the additional sets of "isospectral" flows act on the constituent (adjoint) eigenfunctions as (cf. (113)):

$$\begin{aligned} \partial/\partial \overset{(k)}{t}_n \Phi_i^{(m)} &= \mathcal{M}_{E_k}^{(n)}(\Phi_i^{(m)})\,, \\ \partial/\partial \overset{(k)}{t}_n \Psi_i^{(m)} &= -(\mathcal{M}_{E_k}^{(n)})^*(\Psi_i^{(m)}) \quad \text{for } i \neq k\,, \end{aligned} \tag{138}$$

$$\begin{aligned} \partial/\partial \overset{(k)}{t}_n \Phi_k^{(m)} &= \mathcal{M}_{E_k}^{(n)}(\Phi_k^{(m)}) - \Phi_k^{(n+m)}\,, \\ \partial/\partial \overset{(k)}{t}_n \Psi_k^{(m)} &= -(\mathcal{M}_{E_k}^{(n)})^*(\Psi_k^{(m)}) + \Psi_k^{(n+m)}\,, \end{aligned} \tag{139}$$

with $i,k=1,\dots,M$ and using shorthand notations (110). Such extended KP hierarchies have been previously proposed in Refs. 42. As shown in Ref. 42, we can identify the extended set of "isospectral" flows (136) with the set of isospectral flows $\{\overset{(\ell)}{t}_n\}_{n=1,2,\ldots}^{\ell=1,\ldots,M+1}$ of the (unconstrained) $M+1$-component matrix KP hierarchy. The latter is defined in terms of the $M+1 \times M+1$ matrix Hirota bilinear identities (see Refs. 42):

$$\sum_{k=1}^{M+1} \varepsilon_{ik}\varepsilon_{jk} \int d\lambda\, \lambda^{\delta_{ik}+\delta_{jk}-2} e^{\xi\left(\overset{(k)}{t} - \overset{(k)}{t'},\lambda\right)} \times \tau_{ik}\Big(\ldots,\overset{(k)}{t} - [\lambda^{-1}],\ldots\Big)\tau_{kj}\Big(\ldots,\overset{(k)}{t'} + [\lambda^{-1}],\ldots\Big) = 0\,, \tag{140}$$

which are obeyed by a set of $M(M+1)+1$ tau-functions $\{\tau_{ij}\}$ expressed in terms of the single tau-function τ and the "positive" symmetry flow generating (adjoint) eigenfunctions (134) in the original one-component (scalar) KP hierarchy (89)–(93) as follows:

$$\tau_{11} = \tau_{ii} = \tau\,, \quad \tau_{1i} = \tau\Phi_{i-1}^{(1)}\,, \quad \tau_{i1} = -\tau\Psi_{i-1}^{(1)}\,, \tag{141}$$

$$\tau_{ij} = \varepsilon_{ij}\tau\partial^{-1}(\Phi_{j-1}^{(1)}\Psi_{i-1}^{(1)})\,, \quad i \neq j\,, \quad i,j = 2,\ldots,M+1\,. \tag{142}$$

Here $\varepsilon_{ij} = 1$ for $i \leq j$ and $\varepsilon_{ij} = -1$ for $i > j$, and δ_{ij} are the usual Kronecker symbols.

The above construction of multicomponent (matrix) KP hierarchies out of ordinary one-component ones can be straighforwardly carried over to the case of constrained KP models (100):

$$\partial/\partial t_n \mathcal{L} = \left[\left(\mathcal{L}^{\frac{n}{R}}\right)_+, \mathcal{L}\right], \quad \partial/\partial \overset{(k)}{t}_n \mathcal{L} = [\mathcal{M}_{E_k}^{(n)}, \mathcal{L}]\,, \quad k = 2,\ldots,M+1\,, \tag{143}$$

using the identification (110) for the symmetry-generating (adjoint) eigenfunctions. In this case, however, there is a linear dependence among the flows (136):

$$\sum_{k=2}^{M+1} \partial/\partial \overset{(k)}{t}_n = \delta_{A=\mathbb{1}}^{(n)} = -\frac{\partial}{\partial t_{nR}}\,, \tag{144}$$

therefore, the associated $c\mathrm{KP}_{R,M}$-based extended hierarchy (143) is equivalent to $M \times M$ matrix constrained KP hierarchy.

Similarly, we can start with the subset of "negative" symmetry flows $\delta_{E_k}^{(-n)}$ (117) for $c\mathrm{KP}_{R,M}$ hierarchy:

$$\delta_{E_k}^{(-n)} \equiv \partial/\partial \overset{(k)}{t}_{-n}\,, \quad \mathcal{M}_{E_k}^{(-n)} = \sum_{s=1}^{n} \Phi_k^{(-n-1+s)} D^{-1} \Psi_k^{(-s)}\,, \qquad k = 2,\ldots,M+R\,, \quad n = 1,2,\ldots\,. \tag{145}$$

The flow $\delta_{E_k}^{(-n)}$ for $k=1$ is excluded since $\sum_{k=1}^{M+R} \delta_{E_k}^{(-n)} = \delta_{\mathcal{A}=\mathbb{1}}^{(-n)}$ which vanishes identially as explained in the previous subsection.

The flows (145) also span an infinite-dimensional Abelian algebra commuting with the isospectral flows. Using (145) we now construct another extended KP-type hierarchy analogous to (137)–(139) and based on $c\mathrm{KP}_{R,M}$ (100):

$$\partial/\partial t_n \mathcal{L} = [(\mathcal{L}^{\frac{n}{R}})_+, \mathcal{L}]\,, \qquad \partial/\partial \overset{(k)}{t}_{-n} \mathcal{L} = [\mathcal{M}^{(-n)}_{E_k}, \mathcal{L}]\,, \tag{146}$$

$$\partial/\partial \overset{(k)}{t}_{-n} \Phi^{(-m)}_a = \mathcal{M}^{(-n)}_{E_k}(\Phi^{(-m)}_a)\,, \quad \partial/\partial \overset{(k)}{t}_{-n} \Psi^{(-m)}_a = -(\mathcal{M}^{(-n)}_{E_k})^*(\Psi^{(-m)}_a) \tag{147}$$

for $a \neq k$,

$$\begin{aligned} \partial/\partial \overset{(k)}{t}_{-n} \Phi^{(-m)}_k &= \mathcal{M}^{(-n)}_{E_k}(\Phi^{(-m)}_k) - \Phi^{(-n-m)}_k\,, \\ \partial/\partial \overset{(k)}{t}_{-n} \Psi^{(-m)}_k &= -(\mathcal{M}^{(-n)}_{E_k})^*(\Psi^{(-m)}_k) + \Psi^{(-n-m)}_k\,, \end{aligned} \tag{148}$$

where now $a = 1, \ldots, M+R$, $k = 2, \ldots, M+R$ and we have used shorthand notations (116).

Then, following the steps of our construction in Ref. 42 we arrive at $(M+R)$-component constrained KP hierarchy given in terms of $(M+R)(M+R-1)+1$ tau-functions $\{\tilde{\tau}_{ab}\}$ obeying the corresponding $(M+R)\times(M+R)$ matrix Hirota bilinear identities (cf. (140)). The latter tau-functions are expressed in terms of the original single tau-function τ and the "negative" symmetry flow generating (adjoint) eigenfunctions (116) in the original ordinary $c\mathrm{KP}_{R,M}$ hierarchy (100) as follows:

$$\tilde{\tau}_{11} = \tilde{\tau}_{aa} = \tau\,, \quad \tilde{\tau}_{1a} = \tau L_M(\bar{\varphi}_a)\,, \quad \tilde{\tau}_{a1} = -\tau\bar{\psi}_a\,, \tag{149}$$

$$\tilde{\tau}_{ab} = \varepsilon_{ab}\tau\partial^{-1}(L_M(\bar{\varphi}_b)\bar{\psi}_a)\,, \quad a \neq b\,, \quad a, b = 2, \ldots, M+R\,. \tag{150}$$

Let us note that there exist alternative representation of (constrained) multi-component KP hierarchies based on matrix generalization of Sato pseudo-differential operator formalism[33] (see also Ref. 43). The construction in this section of $c\mathrm{KP}_{R,M}$-based extended (multicomponent KP) hierarchies (143) and (146) has an advantage over the matrix Sato formulation since it allows us to employ the well-known Darboux–Bäcklund techniques from ordinary one-component (scalar) KP hierarchies (full or constrained) in order to obtain new soliton-like solutions of multicomponent (matrix) KP hierarchies (see last Ref. 42 and, especially, Sec. 8 below).

6.5. *Higher-dimensional nonlinear evolution equations as symmetry flows of $c\mathrm{KP}_{R,M}$ hierarchies*

Let us recall that multicomponent (matrix) KP hierarchies (140) contain various physically interesting nonlinear systems such as Davey–Stewartson and N-wave systems, which now can be written entirely in terms of objects belonging to ordinary one-component (constrained) KP hierarchy. Thereby the lowest-grade additional symmetry flow parameters acquire the meaning of coordinates for additional space dimensions.

For instance, the N-wave resonant system ($N = M(M+1)/2$) is given by:

$$\partial_k f_{ij} = f_{ik} f_{kj} \,, \quad i \neq j \neq k \,, \quad i,j,k = 1,\ldots,M+1 \,, \tag{151}$$

$$\partial_k \equiv \partial/\partial \overset{(k)}{t}_1 \,, \quad f_{1i} \equiv \Phi^{(1)}_{i-1} \,, \quad f_{i1} \equiv -\Psi^{(1)}_{i-1} \,, \tag{152}$$

$$f_{ij} \equiv \varepsilon_{ij} \partial^{-1}\left(\Phi^{(1)}_{j-1} \Psi^{(1)}_{i-1}\right) \,, \quad i \neq j \,, \quad i,j = 2,\ldots M+1 \,. \tag{153}$$

As a further example, let us demonstrate in some detail that the well-known Davey–Stewartson system[44] arises as particular subset of symmetry flow equations obeyed by any pair of adjoint eigenfunctions (Φ_i, Ψ_i) ($i =$ fixed) or $(L_M(\bar{\varphi}_a), \bar{\psi}_a)$ ($a =$ fixed). The derivation for (Φ_i, Ψ_i) ($i =$ fixed) has already been presented in Ref. 42. Here for simplicity we take $c\mathrm{KP}_{1,M}$ hierarchy (192) (the general case for $c\mathrm{KP}_{R,M}$ hierarchy (100) is a straightforward generalization of the formulas below) and consider a pair of "negative" symmetry flow generating (adjoint) eigenfunctions $(\phi \equiv L_M(\bar{\varphi}_a), \psi \equiv \bar{\psi}_a)$ ($a =$ fixed), which obeys the following subset of flow equations — w.r.t. $\partial/\partial t_2$, $\bar{\partial} \equiv \partial/\partial \overset{(a)}{t}_{-1}$ and $\partial/\partial \bar{t}_2 \equiv \partial/\partial \overset{(a)}{t}_{-2}$ (cf. Eqs. (119)):

$$\frac{\partial}{\partial t_2}\phi = \left(\partial^2 + 2\sum_{i=1}^{M} \Phi_i \Psi_i\right)\phi \,, \quad \frac{\partial}{\partial t_2}\psi = -\left(\partial^2 + 2\sum_{i=1}^{M} \Phi_i \Psi_i\right)\psi \,, \tag{154}$$

$$\bar{\partial}\phi = \mathcal{M}^{(-1)}(\phi) - \mathcal{L}^{-1}(\phi) \,, \qquad \bar{\partial}\psi = -(\mathcal{M}^{(-1)})^*(\psi) + (\mathcal{L}^{-1})^*(\psi) \,, \tag{155}$$

$$\frac{\partial}{\partial \bar{t}_2}\phi = \mathcal{M}^{(-2)}(\phi) - \mathcal{L}^{-2}(\phi) \,, \qquad \frac{\partial}{\partial \bar{t}_2}\psi = -(\mathcal{M}^{(-2)})^*(\psi) + (\mathcal{L}^{-2})^*(\psi) \,, \tag{156}$$

where

$$\mathcal{M}^{(-1)} \equiv \phi D^{-1}\psi \,, \quad \mathcal{M}^{(-2)} \equiv \mathcal{L}^{-1}(\phi) D^{-1}(\psi) + \phi D^{-1}(\mathcal{L}^{-1})^*(\psi) \,. \tag{157}$$

Using (155) we can rewrite Eqs. (156) as purely differential equation w.r.t. $\bar{\partial}$:

$$\frac{\partial}{\partial \bar{t}_2}\phi = [-\bar{\partial}^2 + 2\bar{\partial}(\partial^{-1}(\phi\psi))]\phi \,, \qquad \frac{\partial}{\partial \bar{t}_2}\psi = [\bar{\partial}^2 - 2\bar{\partial}(\partial^{-1}(\phi\psi))]\psi \,. \tag{158}$$

Now, introducing new time variable $T = t_2 - \bar{t}_2$ and the short-hand notation $Q \equiv \sum_{i=1}^{M} \Phi_i \Psi_i - 2(\phi\psi) - 2\bar{\partial}(\partial^{-1}(\phi\psi))$, and subtracting Eqs. (158) from Eqs. (154), we arrive at the following system of $(2+1)$-dimensional nonlinear evolution equations:

$$\frac{\partial}{\partial T}\phi = \left[\frac{1}{2}(\partial^2 + \bar{\partial}^2) + Q + 2\phi\psi\right]\phi \,, \tag{159}$$

$$\frac{\partial}{\partial T}\psi = -\left[\frac{1}{2}(\partial^2 + \bar{\partial}^2) + Q + 2\phi\psi\right]\psi \,, \tag{160}$$

$$\partial\bar{\partial}Q + (\partial + \bar{\partial})^2(\phi\psi) = 0 \,, \tag{161}$$

which is precisely the standard Davey–Stewartson system[44] for the "negative" (adjoint) eigenfunction pair $(\phi \equiv L_M(\bar{\varphi}_a), \, \psi \equiv \bar{\psi}_a)$ ($a =$ fixed).

7. Algebraic (Drinfeld–Sokolov) Formulation of $c\mathrm{KP}_{R,M}$ Hierarchies and Its Relation to the Sato Formulation

7.1. *Symmetry flows in algebraic (Drinfeld–Sokolov) setting*

Let us very briefly recall the basics of the algebraic (generalized) Drinfeld–Sokolov construction of integrable hierarchies.[45] The main ingredients are:

- Loop (or Kac–Moody) algebra with integral grading $\widehat{\mathcal{G}} = \oplus_{n\in\mathbb{Z}}\mathcal{G}^{(n)}$;
- Fixed semisimple element $E \in \mathcal{G}$ of positive grade, e.g. of grade 1 ($E \equiv E^{(1)}$ – the case to be considered below), meaning that $\widehat{\mathcal{G}}$ splits in a direct sum (as vector space) $\widehat{\mathcal{G}} = \mathcal{K} \oplus \mathcal{M}$ where $\mathcal{K} \equiv \mathrm{Ker}(ad(E))$ and $\mathcal{M} \equiv \mathrm{Im}(ad(E))$ with $[\mathcal{K},\mathcal{K}] \subset \mathcal{K}$, $[\mathcal{K},\mathcal{M}] \subset \mathcal{M}$;
- Dynamical field $A \in \mathcal{M}$ of nonnegative grade smaller than the grade of E, e.g. of grade 0 ($A \equiv A^{(0)}$ – the case to be considered below).

The next basic object is the transfer (monodromy) matrix T taking values in the corresponding group $\widehat{G}$ and satisfying the linear (matrix-Lax) equation:

$$(\partial + E^{(1)} + A^{(0)})T = 0\,. \tag{162}$$

There are two types of relevant boundary conditions for T. The first one, called "regular boundary conditions," sets $T|_{x=x_0} = \mathbb{1}$ at some fixed point $x = x_0$ and the corresponding solution $\mathcal{T} \equiv T_{\mathrm{reg}}$ of (162) reads:

$$\mathcal{T} = P\exp\left\{-\int_{x_0}^{x} dx'(E^{(1)} + A^{(0)}(x'))\right\}\,, \tag{163}$$

which implies that $\mathcal{T} \equiv T_{\mathrm{reg}}$ contains only nonnegative-grade terms. The second type of boundary conditions for the transfer-matrix is called "asymptotic boundary conditions" and the corresponding solution $T \equiv T_{\mathrm{asy}}$ of (162) has, up to an explicit factor, asymptotic expansion in negative grades only:[46]

$$T = \Theta e^{-xE^{(1)}} = USe^{-xE^{(1)}}\,. \tag{164}$$

The group-valued factors U and S in (164):

$$U = \exp\left\{\sum_{j=1}^{\infty} u^{(-j)}\right\}\,, \quad u^{(-j)} \in \mathcal{M}^{(-j)}\,, \tag{165}$$

$$S = e^{\mathfrak{s}}\,, \quad \mathfrak{s} = \sum_{j=1}^{\infty} \mathfrak{s}^{(-j)} \in \mathcal{K} \tag{166}$$

have a well-defined meaning of "gauge-rotating" the matrix Lax operator in (162) to the "bare" one:

$$US(\partial + E^{(1)})S^{-1}U^{-1} = \partial + E^{(1)} + A^{(0)}\,. \tag{167}$$

Symmetry flows in the algebraic (generalized Drinfeld–Sokolov) formalism are given as flows acting on $T \equiv T_{\rm asy}$ (164) and $\mathcal{T} \equiv T_{\rm reg}$ (163), which are defined in terms of dressing of constant algebraic elements $X^{(\pm n)}$ of positive or negative grades:[47,48]

$$\delta_X^{(n)} T = (TX^{(n)}T^{-1})_- T\,, \qquad \delta_X^{(n)} \mathcal{T} = -(TX^{(n)}T^{-1})_+ \mathcal{T} \quad n \geq 1\,, \tag{168}$$

with $X^{(n)} \in \mathcal{K}$ (for $X^{(n)} \in \mathcal{M}$ the flows (168) are not well-defined[e]), and

$$\delta_X^{(-n)} \mathcal{T} = (\mathcal{T}X^{(-n)}\mathcal{T}^{-1})_+ \mathcal{T}\,, \qquad \delta_X^{(-n)} T = -(\mathcal{T}X^{(-n)}\mathcal{T}^{-1})_- T \quad n \geq 1\,, \tag{169}$$

where $X^{(-n)}$ may be arbitrary negative-grade element.

Applying the flows on the basic linear problem Eq. (162) and using the fact that

$$[\partial + E^{(1)} + A^{(0)}, TXT^{-1}] = 0\,, \qquad [\partial + E^{(1)} + A^{(0)}, \mathcal{T}X\mathcal{T}^{-1}] = 0\,, \tag{170}$$

for any X, one finds for the flow action of the dynamical field:

$$\begin{aligned} \delta_X^{(n)} A^{(0)} &= [\partial + E^{(1)} + A^{(0)}, (TX^{(n)}T^{-1})_+] \\ &= -[E^{(1)}, (TX^{(n)}T^{-1})_{(-1)}]\,, \end{aligned} \tag{171}$$

$$\begin{aligned} \delta_X^{(-n)} A^{(0)} &= [\partial + E^{(1)} + A^{(0)}, (\mathcal{T}X^{(-n)}\mathcal{T}^{-1})_-] \\ &= [E^{(1)}, (\mathcal{T}X^{(-n)}\mathcal{T}^{-1})_{(-1)}]\,, \end{aligned} \tag{172}$$

where the subscript (-1) indicates taking the grade $= -1$ part.

The isospectral flows are special subset of positive-grade flows (168) corresponding to dressing of the elements $E^{(n)}$ of the positive-grade part of the center of $\mathcal{K}$, i.e.:

$$\frac{\partial}{\partial t_n} \equiv \delta_{E^{(n)}}^{(n)}\,, \quad \left(\frac{\partial}{\partial t_n} - (TE^{(n)}T^{-1})_-\right) T = 0\,, \quad \text{for } n \geq 2\,. \tag{173}$$

Using the explicit form of the flows Eqs. (168) and (169) one can show (cf. Ref. 48) that

- The positive-grade flows (168) span an algebra isomorphic to the "bare" algebra $\mathcal{K}_+$:

$$[\delta_X^{(n)}, \delta_Y^{(m)}] T = (T[X^{(n)}, Y^{(m)}]T^{-1})_- T = \delta_{[X,Y]}^{(n+m)} T\,. \tag{174}$$

  In particular, they commute with the isospectral flows (173) which justifies their name of symmetry flows.

[e] This ill-definiteness is due to the right-most exponential factor in $T \equiv T_{\rm asy}$ (164) carrying an infinite positive-grade tail of terms. Thus if $X^{(n)} \in \mathcal{M}$, then in $TX^{(n)}T^{-1} = \Theta e^{-xE^{(1)}} X^{(n)} e^{xE^{(1)}} \Theta^{-1}$ the middle factor on the r.h.s. $e^{-xE^{(1)}} X^{(n)} e^{xE^{(1)}}$ is an infinite series in positive-grade terms, whereas by construction $\Theta = US$ (164)–(166) is an infinite series in negative-grade terms. Therefore, $(TX^{(n)}T^{-1})_-$ for $X^{(n)} \in \mathcal{M}$ is ill-defined as loop-algebra element, since any fixed-grade term in the grade expansion of the latter will be given as an infinite sum.

- The negative-grade flows (169) span an algebra isomorphic to $\widehat{\mathcal{G}}_-$ — the negative-grade part of the underlying loop algebra $\widehat{\mathcal{G}}$:

$$[\delta_X^{(-n)}, \delta_Y^{(-m)}]\mathcal{T} = (\mathcal{T}[X^{(-n)}, Y^{(-m)}]\mathcal{T}^{-1})_+\mathcal{T} = \delta_{[X,Y]}^{(-n-m)}\mathcal{T}\,. \tag{175}$$

- Negative-grade flows commute with positive-grade flows:

$$[\delta_X^{(n)}, \delta_Y^{(-m)}]\mathcal{T} = 0\,. \tag{176}$$

In particular, negative flows commute with the isospectral flows (173) and, therefore, the negative-grade flows are similarly symmetry flows.

$T \equiv T_{\text{asy}}$ and $\mathcal{T} \equiv T_{\text{reg}}$ are generating functionals of *local* and *nonlocal* conserved charges of the underlying integrable hierarchy,[29,49] respectively, where conservation is understood w.r.t. the isospectral flows (173).

In what follows it will be more convenient to use an object $\tilde{T}$ related to $T \equiv T_{\text{asy}}$ (164) as:

$$\tilde{T} = \Theta \exp\left\{-xE^{(1)} - \sum_{\ell\geq 2} t_\ell E^{(\ell)}\right\} \to \left(\frac{\partial}{\partial t_n} + (\tilde{T}E^{(n)}\tilde{T}^{-1})_+\right)\tilde{T} = 0\,. \tag{177}$$

Using $\tilde{T}$ one can extend the definition of positive-grade symmetry flows (168) to construct also the Virasoro additional symmetry flows within the algebraic (generalized Drinfeld–Sokolov) approach. For instance, in the case of $\widehat{\mathcal{G}} = \widehat{\text{SL}}(M+1)$ with standard homogeneous grading $Q = \lambda\partial/\partial\lambda$, which is the algebraic description of $c\text{KP}_{1,M}$ hierarchies (see the next subsection), the Virasoro symmetry flows are given as (cf. Ref. 48[f]):

$$\delta_{\text{Vir}}^{(n)}\tilde{T} = (\tilde{T}\ell_n\tilde{T}^{-1})_-\tilde{T}\,, \qquad \ell_n \equiv -\lambda^{n+1}\partial/\partial\lambda \equiv -\lambda^n Q\,, \tag{178}$$

whereas in the more general case of $\widehat{\mathcal{G}} = \widehat{\text{SL}}(M+R)$ with nonstandard grading Q_R (Eq. (208) below) the Virasoro flows exists for integer values of the modes modulo R:[48]

$$\delta_{\text{Vir}}^{(nR)}\tilde{T} = (\tilde{T}\tilde{\ell}_{nR}\tilde{T}^{-1})_-\tilde{T}\,, \qquad \tilde{\ell}_{nR} \equiv -\lambda^n Q_R\,. \tag{179}$$

In terms of the original transfer matrix $T \equiv T_{\text{asy}}$ (164) the flow Eqs. (179) amount to "dressing" of isospectral-time-dependent "bare" Virasoro generators:[48]

$$\delta_{\text{Vir}}^{(nR)}T = \left(T\left(\tilde{\ell}_{nR} - \sum_{k=2}^{\infty} kt_k E^{(k+nR)}\right)T^{-1}\right)_- T\,, \tag{180}$$

and similarly for (178) (where $R = 1$).

[f] For (m)KdV hierarchies, which are the limiting case $M = 0$ of the class of $c\text{KP}_{R,M}$ hierarchies (100), Virasoro additional symmetries via dressing within the algebraic Drinfeld–Sokolov approach have been considered in Ref. 50.

Let us note that the full Virasoro algebra of the additional symmetries for constrained $c\mathrm{KP}_{R,M}$ integrable hierarchies (100) has been first constructed in Refs. 34 and 24 within the Sato pseudo-differential approach. This construction involves a nontrivial modification of ordinary Orlov–Schulmann flows[51,18,52] generating Virasoro and $\mathbf{W_{1+\infty}}$ symmetries of the general unconstrained KP hierarchy. Orlov–Schulman flows do not generate symmetries for constrained $c\mathrm{KP}_{R,M}$ hierarchies since they do not preserve the constrained form of the pertinent Sato Lax operator (100).

Now, let us consider the system of equations consisting of the basic linear problem (162) for $\mathcal{T} = T_{\mathrm{reg}}$ (163) and the lowest negative-grade flow equation (first Eq. (169)) with $X^{(-1)} \equiv E^{(-1)}$ belonging to the center of $\mathcal{K} = \mathrm{Ker}(ad(E^{(1)}))$ (denoting $\delta^{(-1)}_{E^{(-1)}} \equiv -\bar{\partial}$):

$$(\partial + E^{(1)} + A^{(0)})\mathcal{T} = 0\,, \qquad (\bar{\partial} + (\mathcal{T}E^{(-1)}\mathcal{T}^{-1})_+)\mathcal{T} = 0\,. \tag{181}$$

$\mathcal{T}$ has a grading representation:

$$\mathcal{T} = \mathcal{T}^{(0)}e^{\Omega}\,, \qquad \Omega \equiv \sum_{k=1}^{\infty} \omega^{(k)}\,. \tag{182}$$

The zero-grade orders of both Eqs. (181) yield accordingly:

$$(\partial + A^{(0)})\mathcal{T}^{(0)} = 0 \quad \text{or} \quad \partial\mathcal{T}^{(0)}\mathcal{T}^{(0)-1} = -A^{(0)} \to \mathcal{T}^{(0)} = P\exp\left\{-\int^x dx' A^{(0)}(x')\right\}\,, \tag{183}$$

$$(\bar{\partial} + \mathcal{T}^{(0)}[\omega^{(1)}, E^{(-1)}]\mathcal{T}^{(0)-1})\mathcal{T}^{(0)} = 0 \to \mathcal{T}^{(0)-1}\bar{\partial}\mathcal{T}^{(0)} = -[\omega^{(1)}, E^{(-1)}]\,. \tag{184}$$

In particular, we obtain the explicit $\bar{\partial}$-flow equation for A:

$$\bar{\partial}A^{(0)} = -[E^{(1)}, \mathcal{T}^{(0)}E^{(-1)}\mathcal{T}^{(0)-1}]\,. \tag{185}$$

It is straightforward to check, using (181), (183) and (185), that the zero-curvature condition:

$$[\bar{\partial} + (\mathcal{T}E^{(-1)}\mathcal{T}^{-1})_+, \partial + E^{(1)} + A^{(0)}] = 0 \tag{186}$$

is identically satisfied. Note, that due to the first Eq. (181), Eq. (186) can be equivalently written entirely in terms of $\mathcal{T}^{(0)}$ as:

$$[\bar{\partial} - \mathcal{T}^{(0)}E^{(-1)}\mathcal{T}^{(0)-1}, \partial + E^{(1)} + A^{(0)}] = 0\,. \tag{187}$$

Now, let us observe that substituting $A^{(0)} = -\partial\mathcal{T}^{(0)}\mathcal{T}^{(0)-1}$ (according to (183)) in Eq. (185), the latter acquires the form:

$$\bar{\partial}(\partial\mathcal{T}^{(0)}\mathcal{T}^{(0)-1}) - [E^{(1)}, \mathcal{T}^{(0)}E^{(-1)}\mathcal{T}^{(0)-1}] = 0\,, \tag{188}$$

together with the constraints:

$$\partial\mathcal{T}^{(0)}\mathcal{T}^{(0)-1}|_{\mathcal{H}}(= -A^{(0)}|_{\mathcal{H}}) = 0\,, \quad \text{since} \quad A^{(0)} \in \mathcal{M}\,, \tag{189}$$

$$\mathcal{T}^{(0)-1}\bar{\partial}\mathcal{T}^{(0)}|_{\mathcal{H}}(= [E^{(-1)}, \omega^{(1)}]|_{\mathcal{H}}) = 0\,, \quad \text{since} \quad [E^{(-1)}, \omega^{(1)}] \in \mathcal{M}\,. \tag{190}$$

Setting $g \equiv \mathcal{T}^{(0)}$ in Eq. (188) the latter is identified with Eq. (86) with $E_{+}^{(1)} \equiv E^{(1)}$ and $E_{-}^{(1)} \equiv E^{(-1)}$ (recall that $\delta^{(-1)}_{E^{(-1)}} \equiv -\bar{\partial}$). Moreover, the constraints (189)–(190) are particular cases of the general constraints on the WZNW "currents" (87)–(88). Therefore, we conclude that the field equations of motion (in "light-cone" coordinates) of the gauged G/H WZNW model can be identified as a particular case of symmetry flow equations of a generalized Drinfeld–Sokolov integrable hierarchy based on the loop algebra $\widehat{\mathcal{G}}$ and where the subalgebra $\widehat{\mathcal{H}}$ is such that $\widehat{\mathcal{H}} = \mathrm{Ker}(ad(E_{+}^{(1)}))$, where $E_{+}^{(1)}$ is semisimple element and both $E_{\pm}^{(\pm 1)}$ belong to the center of $\widehat{\mathcal{H}}$.

In Ref. 53 it was shown that Eq. (188) in the special case of $\widehat{\mathrm{SL}}(2)$ with $E^{(1)} = \frac{\lambda}{2}\sigma_3$, which corresponds in Sato formalism to $c\mathrm{KP}_{1,1}$ hierarchy (Eq. (100) with $R = 1$ and $M = 1$), contains the complex Sine–Gordon equations. Also, in Ref. 53 the more general Drinfeld–Sokolov hierarchies based on $\mathcal{G} = \widehat{\mathrm{SL}}(R)$ with:

$$E^{(1)} = -\left(\sum_{i=1}^{R-1} E^{(1)}_{\alpha_i} + E^{(0)}_{-(\alpha_1+\cdots+\alpha_{R-1})}\right), \quad A^{(0)} = -\sum_{i=1}^{R-1}(\partial\varphi_1 + \cdots + \partial\varphi_i)H_{\alpha_i} \tag{191}$$

(cf. Eq. (209) below) has been considered, which corresponds in Sato formalism to the limiting case $c\mathrm{KP}_{R,0}$ of $c\mathrm{KP}_{R,M}$ hierarchies (Eq. (100) with $M = 0$, i.e. the purely differential mKdV Lax operator). It has been shown in Ref. 53 that Eqs. (188) in the latter special case reduce to the equations of motion of affine Toda field theories. In what follows we extend this discussion to the whole class of $c\mathrm{KP}_{R,M}$ hierarchies.

7.2. *Algebraic (Drinfeld–Sokolov) formulation of $c\mathbf{KP}_{R,M}$ hierarchies*

We start with the subclass of $\mathbf{cKP}_{1,M}$ hierarchies (cf. (100)) with Sato Lax operator:

$$\mathcal{L}_{1,M} = D + \sum_{i=1}^{M} \Phi_i D^{-1} \Psi_i\,. \tag{192}$$

Let us introduce the column vector:

$$\begin{pmatrix} \psi_1 \\ \vdots \\ \psi_M \\ \psi_{M+1} \end{pmatrix} = e^{-\frac{1}{M+1}\xi(t,\lambda)} \begin{pmatrix} S_M \\ \vdots \\ S_1 \\ \psi_{\mathrm{BA}} \end{pmatrix}, \tag{193}$$

where the following shorthand notations are used (recall (98)):

$$\begin{aligned} S_i(t,\lambda) &\equiv \partial^{-1}(\psi_{\rm BA}(t,\lambda)\Psi_i(t)) = \frac{1}{\lambda}\psi_{\rm BA}(t,\lambda)\Psi_i(t-[\lambda^{-1}])\,, \\ S_i^*(t,\lambda) &\equiv \partial^{-1}(\psi^*_{\rm BA}(t,\lambda)\Phi_i(t)) = -\frac{1}{\lambda}\psi^*_{\rm BA}(t,\lambda)\Phi_i(t+[\lambda^{-1}])\,, \end{aligned} \tag{194}$$

with $i=1,\ldots,M$. It is well-known[54] that the linear spectral Lax equation (first Eq. (91)) for the BA wave function $\psi_{\rm BA}$ of (192) can be represented in the following algebraic (generalized Drinfeld–Sokolov) form:

$$[D+E+A]\begin{pmatrix} \psi_1 \\ \vdots \\ \psi_M \\ \psi_{M+1} \end{pmatrix} = 0\,, \tag{195}$$

with

$$\begin{aligned} E \equiv E^{(1)} &= \frac{\lambda}{M+1}\begin{pmatrix} 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \\ 0 & \cdots & 0 & -M \end{pmatrix} \equiv H^{(1)}_{\lambda_M}\,, \\ A \equiv A^{(0)} &= \begin{pmatrix} 0 & \cdots & 0 & -\Psi_M \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & -\Psi_1 \\ \Phi_M & \cdots & \Phi_1 & 0 \end{pmatrix}. \end{aligned} \tag{196}$$

Here the underlying loop algebra is $\widehat{\rm SL}(M+1)$ with standard homogeneous gradation $Q=\lambda\partial/\partial\lambda$ and the corresponding kernel $\mathcal{K}\equiv{\rm Ker}(ad(E))$ and image $\mathcal{M}\equiv{\rm Im}(ad(E))$ are given by

$$\mathcal{K} = \left\{E^{(n)} \equiv H^{(n)}_{\lambda_M}, H^{(n)}_1,\ldots,H^{(n)}_{M-1}, E^{(n)}_{\pm(\alpha_{k_1}+\cdots+\alpha_{k_s})}\right\}_{n\in\mathbb{Z}}\,, \tag{197}$$

$$\mathcal{M} = \left\{E^{(n)}_{\pm\alpha_M}, E^{(n)}_{\pm(\alpha_{k_1}+\cdots+\alpha_{k_s}+\alpha_M)}\right\}_{n\in\mathbb{Z}}\,. \tag{198}$$

λ_M is the last ${\rm SL}(M+1)$ fundamental weight, $1\le k_1\le\cdots\le k_s\le M-1$ and $s=1,\ldots,M-1$. The center of $\mathcal{K}$ generating the isospectral flows via (173) is $\mathcal{C}(\mathcal{K})=\{E^{(n)}\equiv H^{(n)}_{\lambda_M}\}_{n\in\mathbb{Z}}$.

In order to establish the equivalence between the algebraic and Sato formulations, we need to establish the opposite transition, i.e. the transition from the basic object in the algebraic framework — the transfer matrix (162), to the objects characterizing the integrable hierarchy in Sato formalism. This transition is provided by the following formula (cf. definition (193)):

$$\begin{pmatrix} S_M \\ \vdots \\ S_1 \\ \psi_{\rm BA} \end{pmatrix} \equiv e^{\frac{1}{M+1}\xi(t,\lambda)} \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_M \\ \psi_{M+1} \end{pmatrix} = e^{\frac{1}{M+1}\xi(t,\lambda)} \tilde{T} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} = e^{\xi(t,\lambda)} \Theta \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \tag{199}$$

and similarly for the adjoint objects:

$$\begin{pmatrix} S^*_M \\ \vdots \\ S^*_1 \\ \psi^*_{\rm BA} \end{pmatrix} = e^{-\frac{1}{M+1}\xi(t,\lambda)} \tilde{T}^{*-1} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} = e^{-\xi(t,\lambda)} \Theta^{*-1} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \tag{200}$$

where $\tilde{T}$ is the "asymptotic" transfer matrix (177). The derivation of (199)–(200) uses the special grading properties of the constituent group factors in $\tilde{T}$ (177) and compares them with the λ dependence of $\psi^{(*)}_{\rm BA}(t,\lambda)$ (92) and $S^{(*)}_i(t,\lambda)$ (98).

The group factor S in the decomposition of $\tilde{T}$ (177) has the form:

$$S = e^{\mathfrak{s}(\lambda)}, \quad \mathfrak{s}(\lambda) = \sum_{k=1}^{M} \mathfrak{s}_k(\lambda) H_k + \sum_{\beta,\beta\neq\alpha M} \mathfrak{s}_{\pm\beta}(\lambda) E_{\pm\beta}$$
$$\to S \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} = e^{-\mathfrak{s}_M(\lambda)} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}. \tag{201}$$

Accordingly, the other group factor U is of the form:

$$U = \exp \begin{pmatrix} 0 & \cdots & 0 & b_M \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & b_1 \\ a_M & \cdots & a_1 & 0 \end{pmatrix} \to U \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$
$$= \cosh(\sqrt{\mathbf{a}\cdot\mathbf{b}}) \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} + \frac{\sinh(\sqrt{\mathbf{a}\cdot\mathbf{b}})}{\sqrt{\mathbf{a}\cdot\mathbf{b}}} \begin{pmatrix} b_M \\ \vdots \\ b_1 \\ 0 \end{pmatrix}. \tag{202}$$

Thus, for the BA wave functions (92) in the algebraic framework we obtain

$$\psi^{(*)}_{\mathrm{BA}}(t,\lambda) = e^{\pm\xi(t,\lambda)} \frac{\tau(t \mp [\lambda^{-1}])}{\tau(t)}$$
$$= e^{\pm\xi(t,\lambda)} e^{\mp\mathfrak{s}_M(\lambda)} \langle (0,0,\ldots,1)| U^{(*-1)} \left| \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \right\rangle . \tag{203}$$

Equations (203), taking into account (202), imply our main result about the algebraic formalism's expression for the tau-function of $c\mathrm{KP}_{1,M}$ hierarchy (192):

$$\frac{\tau(t-[\lambda^{-1}])}{\tau(t+[\lambda^{-1}])} = e^{-2\mathfrak{s}_M(\lambda)} , \tag{204}$$

where $\mathfrak{s}_M(\lambda)$ is the coefficient in front of H_M in the expansion (201) of $\mathfrak{s}(\lambda)$ in the pertinent $\mathcal{K}$ (197).

Taking the scalar product of both column vectors (199) and (200) we find the following constraint relation:

$$\psi_{\mathrm{BA}}(t,\lambda)\psi^*_{\mathrm{BA}}(t,\lambda) + \sum_{i=1}^{M} S_i(t,\lambda) S^*_i(t,\lambda) = 1 . \tag{205}$$

For the general constraint relation involving BA and SEP functions, valid for any constrained $c\mathrm{KP}_{R,M}$ model, see Eq. (222) below.

In the particular case of $c\mathrm{KP}_{1,1}$ hierarchies, i.e. with Sato Lax operator $\mathcal{L} = D + \Phi D^{-1}\Psi$ which corresponds to a (generalized) Drinfeld–Sokolov hierarchy based on $\widehat{\mathrm{SL}}(2)$ with standard homogeneous grading and

$$E \equiv E^{(1)} = \frac{1}{2}\lambda\sigma_3 , \qquad A \equiv A^{(0)} = \Phi\sigma_- - \Psi\sigma_+ , \tag{206}$$

we can, using (199)–(200) for $M = 1$, explicitly express all matrix elements of the "asymptotic" transfer matrix (164) in terms of Sato-formalism objects:

$$\tilde{T} = \begin{pmatrix} \frac{\tau(t+[\lambda^{-1}])}{\tau(t)} & \frac{1}{\lambda}\Psi(t-[\lambda^{-1}])\frac{\tau(t-[\lambda^{-1}])}{\tau(t)} \\ \frac{1}{\lambda}\Phi(t+[\lambda^{-1}])\frac{\tau(t+[\lambda^{-1}])}{\tau(t)} & \frac{\tau(t-[\lambda^{-1}])}{\tau(t)} \end{pmatrix} e^{\left(-\frac{1}{2}\lambda x - \frac{1}{2}\sum_{j=2}^{\infty}\lambda^j t_j\right)\sigma_3} . \tag{207}$$

The algebraic formulation of the general $c\mathrm{KP}_{R,M}$ constrained KP hierarchies (100) with $R \geq 2$ is given in terms of $\widehat{\mathrm{SL}}(M+R)$ with the following nonstandard grading:

$$Q_R = R\lambda\partial/\partial\lambda + \sum_{i=1}^{R-1} H^{(0)}_{\lambda_{M+i}} \equiv \mu\partial/\partial\mu + \sum_{i=1}^{R-1} H^{(0)}_{\lambda_{M+i}} , \quad \lambda = \mu^R , \tag{208}$$

and the following choice for the fixed semisimple element:

$$E \equiv E^{(1)} = -\left(\sum_{i=1}^{R-1} E^{(1)}_{\alpha_{M+i}} + E^{(0)}_{-(\alpha_{M+1}+\cdots+\alpha_{M+R-1})}\right) . \tag{209}$$

Also, in (208) we introduced a new spectral parameter μ of ordinary grade one for later convenience. The linear problem for the monodromy (transfer) matrix is given by the matrix Lax operator:[54]

$$
\begin{aligned}
L = &\begin{pmatrix}
D & 0 & \cdots & 0 & -\Psi_1 & 0 & \cdots & \cdots & \cdots & 0 \\
0 & D & 0 & \cdots & -\Psi_2 & 0 & \cdots & \cdots & \cdots & 0 \\
\vdots & & \ddots & & \vdots & 0 & \cdots & \cdots & \cdots & \vdots \\
0 & & & D & -\Psi_M & 0 & \cdots & \cdots & \cdots & 0 \\
\varphi_1 & \varphi_2 & \cdots & \varphi_M & D - v_1 & -1 & 0 & \cdots & \cdots & 0 \\
0 & & \cdots & 0 & 0 & D - v_2 & -1 & 0 & \cdots & \vdots \\
0 & & \cdots & 0 & 0 & 0 & D - v_3 & -1 & \cdots & \vdots \\
\vdots & & \cdots & 0 & 0 & \cdots & 0 & \ddots & \ddots & 0 \\
\vdots & & \cdots & 0 & 0 & \cdots & 0 & \ddots & \ddots & -1 \\
0 & & \cdots & 0 & -\lambda & 0 & \cdots & 0 & \cdots & D - v_R
\end{pmatrix} \\
&\equiv D + E + A\,.
\end{aligned}
\tag{210}
$$

The relation between the coefficients in (100) and those in (210) is as follows:

$$
\begin{gathered}
(D - v_R)\cdots(D - v_1) = D^R + \sum_{i=1}^{R-2} u_i D^i\,, \\
v_R \equiv -\sum_{j=1}^{R-1} v_j\,, \quad \Phi_i = (\partial - v_R)\cdots(\partial - v_2)\varphi_i\,.
\end{gathered}
\tag{211}
$$

Recall the splitting of $\mathcal{K} = \mathrm{Ker}(ad(E)) = \oplus_{m\in\mathbb{Z}}\mathcal{K}^{(m)}$ according to the grading (208):[54]

$$
\mathcal{K}^{(nR)} = \left\{\frac{M+R}{M}H^{(n)}_{\lambda_M} \equiv b_{nR}, H^{(n)}_1, \ldots, H^{(n)}_{M-1}, E^{(n)}_{\pm(\alpha_{k_1}+\cdots+\alpha_{k_s})}\right\}, \tag{212}
$$

with the same notations as in Eq. (197), and

$$
\mathcal{K}^{(nR+\ell)} = \left\{\sum_{i=1}^{R-\ell} E^{(n+1)}_{\alpha_{M+i}+\cdots+\alpha_{M+\ell-1+i}} + \sum_{i=1}^{\ell} E^{(n)}_{-(\alpha_{M+i}+\cdots+\alpha_{M+R-1-\ell+i})} \equiv -b_{nR+\ell}\right\}, \tag{213}
$$

where $\ell = 1, \ldots, R-1$. The center of $\mathcal{K}$ generating the isospectral flows according to (173) is $\mathcal{C}(\mathcal{K}) = \{b_n\}_{n\in\mathbb{Z}}$ with b_n as defined in (212)–(213).

Now we are interested in the opposite transition: from the algebraic (Drinfeld–Sokolov) to Sato formulation of $c\mathrm{KP}_{R,M}$ hierarchies (100). This transition is established in a way similar to the simpler case for $c\mathrm{KP}_{1,M}$ hierarchies (cf. Eqs. (199)–(200)):

$$\begin{pmatrix} S_1 \\ \vdots \\ S_M \\ \psi_{\mathrm{BA}} \\ \psi_{M+2} \\ \vdots \\ \psi_{M+R} \end{pmatrix} = T \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ \mu \\ \vdots \\ \mu^{R-1} \end{pmatrix} = e^{\xi(t,\mu)} U S \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ \mu \\ \vdots \\ \mu^{R-1} \end{pmatrix}, \qquad (214)$$

where

$$\psi_{M+j} = \mu^R (\partial - v_j)^{-1} (\partial - v_{j+1})^{-1} \cdots (\partial - v_R)^{-1} \psi_{\mathrm{BA}}(t,\mu), \quad j = 2, \ldots, R, \quad (215)$$

and for the corresponding adjoint quantities:

$$\begin{pmatrix} \tilde{S}_1 \\ \vdots \\ \tilde{S}_M \\ \psi^*_{M+1} \\ \psi^*_{M+2} \\ \vdots \\ \psi^*_{\mathrm{BA}} \end{pmatrix} = T^{*-1} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \mu^{R-1} \\ \mu^{R-2} \\ \vdots \\ 1 \end{pmatrix} = e^{-\xi(t,\mu)} U^{*-1} S^{*-1} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \mu^{R-1} \\ \mu^{R-2} \\ \vdots \\ 1 \end{pmatrix}, \qquad (216)$$

where

$$\psi^*_{M+j} = (-1)^{R-j} (\partial + v_{j+1}) \cdots (\partial + v_R) \psi^*_{\mathrm{BA}}(t,\mu), \quad j = 1, \ldots, R-1, \qquad (217)$$

$$\tilde{S}_i \equiv (-1)^{R-1} \partial^{-1} (\varphi_i \psi^*_{M+1}) = (-1)^{R-1} \partial^{-1} \left(\varphi_i \prod_{j=1}^{R} (\partial + v_j) \psi^*_{\mathrm{BA}} \right). \qquad (218)$$

In obtaining the last relation in (214) use was made of

$$b_N \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ \mu \\ \vdots \\ \mu^{R-1} \end{pmatrix} = -\mu^N \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ \mu \\ \vdots \\ \mu^{R-1} \end{pmatrix}, \qquad (219)$$

with b_N as defined in (212)–(213).

Taking the scalar product of the column vectors (214) and (216) we arrive at the following identity generalizing identity (205):

$$\langle(\psi^*)^T|\psi\rangle \equiv \sum_{i=1}^{M} \tilde{S}_i S_i + \psi^*_{M+1}\psi_{\rm BA} + \sum_{j=2}^{R-1} \psi^*_{M+j}\psi_{M+j} + \psi^*_{\rm BA}\psi_{M+R} = R\mu^{R-1} \,. \quad (220)$$

The same identity can also be directly derived within the Sato pseudo-differential approach by using the relation:[33]

$$\psi^*_{\rm BA}\mathcal{L}_+(\psi_{\rm BA}) - \mathcal{L}^*_+(\psi^*_{\rm BA})\psi_{\rm BA} = \partial\,{\rm Res}(D^{-1}\psi^*_{\rm BA}\mathcal{L}\psi_{\rm BA}D^{-1})\,, \quad (221)$$

and using the explicit splitting of $\mathcal{L} \equiv \mathcal{L}_{R,M}$ (100) into differential and purely pseudo-differential parts. Its form in Sato formalism reads:

$${\rm Res}(D^{-1}\psi^*_{\rm BA}(t,\mu)\mathcal{L}\psi_{\rm BA}(t,\mu)D^{-1}) + \sum_{i=1}^{M} S^*_i(t,\mu)S_i(t,\mu) = R\mu^{R-1}\,. \quad (222)$$

Inserting the expressions for $\psi^{(*)}_{M+j}$ and $\tilde{S}_i$ (Eqs. (215) and (217)–(218)) in the l.h.s. of (220), one can show after some algebra that both forms (220) and (222) coincide.

Remark. The identity (222) (or (220)) is a constraint on the pertinent (adjoint) Baker–Akhiezer functions and can be viewed as alternative definition of the constrained $c\mathrm{KP}_{R,M}$ hierarchy (100).

7.3. *Symmetry flows of* $c\mathbf{KP}_{R,M}$ *hierarchies in the algebraic setting*

Upon inspection of the kernel $\mathcal{K}$ (Eqs. (212)–(213)) we deduce that all $c\mathrm{KP}_{R,M}$ models (100), i.e. those defined in the algebraic (generalized Drinfeld–Sokolov) framework through $\widehat{\mathrm{SL}}(M+R)$ loop algebras with nonstandard gradings (208)–(210), share *the same* $(\widehat{\mathrm{SL}}(M))_+$ algebra of positive-grade additional symmetries irrespective of the value of $R \geq 2$, which is the same as the algebra of positive-grade additional symmetries of the subclass of models $c\mathrm{KP}_{1,M}$ (192)–(198). For the latter models these symmetries are generated via dressing of the positive-grade kernel elements (notations as in (197)):

$$\{H^{(n)}_1, \ldots, H^{(n)}_{M-1}, E^{(n)}_{\pm(\alpha_{k_1}+\cdots+\alpha_{k_s})}\}_{n\geq 1}\,, \quad (223)$$

with the "asymptotic" transfer matrix according to (168). Accordingly, for $c\mathrm{KP}_{R,M}$ hierarchies with $R \geq 2$ the $(\widehat{\mathrm{SL}}(M))_+$ symmetry flows are generated via dressing of the positive-modulo-R-grade kernel elements in $\mathcal{K}^{(nR)}$ (212).

Furthermore, $c\mathrm{KP}_{R,M}$ models within the algebraic framework possess another algebra of additional symmetries $(\widehat{\mathrm{SL}}(M+R))_-$, commuting with $(\widehat{\mathrm{SL}}(M))_+$ symmetry algebra, which is obtained via dressing of the whole negative-modulo-R-grade part of the underlying loop algebra in the generalized Drinfeld–Sokolov scheme with the "regular" transfer (monodromy) matrix according to (169). Therefore, there is a complete agreement of additional symmetries both within the Sato pseudo-differential operator formulation (cf. (125)) and algebraic (generalized Drinfeld–Sokolov) formulation of $c\mathrm{KP}_{R,M}$ hierarchies.

8. Darboux–Bäcklund Transformations and Multiple-Wronskian Solutions of $c\mathbf{KP}_{R,M}$ Hierarchies

After having shown in Subsec. 7.1 that the gauge-fixed equations of motion of gauged WZNW models are additional-symmetry flow equations for generalized Drinfeld–Sokolov hierarchies within the algebraic framework, and after establishing the equivalence between algebraic (generalized Drinfeld–Sokolov) and Sato formulations of the KP-type hierarchies, we can now employ the well-known Darboux–Bäcklund techniques in Sato formalism to generate solutions for gauged WZNW field equations.

Below we will consider explicitly the case of gauged $\mathrm{SL}(M+1)/\mathrm{U}(1)\times\mathrm{SL}(M)$ WZNW models where $E^{(\pm)}_{\pm} = H^{(\pm 1)}_{\lambda_M}$ in the pertinent field equation (86) or, equivalently, equations (188):

$$\bar{\partial}(\partial\mathcal{T}^{(0)}\mathcal{T}^{(0)-1}) - \left[H^{(1)}_{\lambda_M}, \mathcal{T}^{(0)}H^{(-1)}_{\lambda_M}\mathcal{T}^{(0)-1}\right] = 0\,, \tag{224}$$

with the constraints:

$$\partial\mathcal{T}^{(0)}\mathcal{T}^{(0)-1} = -A^{(0)} = -\begin{pmatrix} 0 & \cdots & 0 & -\Psi_M \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & -\Psi_1 \\ \Phi_M & \cdots & \Phi_1 & 0 \end{pmatrix}$$

$$\text{i.e.}\quad \partial\mathcal{T}^{(0)}\mathcal{T}^{(0)-1}|_{\mathrm{U}(1)\oplus\mathrm{SL}(M)} = 0\,, \tag{225}$$

and

$$\mathcal{T}^{(0)-1}\bar{\partial}\mathcal{T}^{(0)}|_{\mathrm{U}(1)\oplus\mathrm{SL}(M)} = 0\,, \tag{226}$$

due to second Eq. (184) (with $E^{(-1)} = H^{(-1)}_{\lambda_M}$). Also recall that $-\bar{\partial} \equiv \delta^{(-1)}_{E^{(-1)}} \equiv \delta^{(-1)}_{H_{\lambda_M}}$ is the lowest negative-grade additional symmetry flow in the algebraic (generalized Drinfeld–Sokolov) formulation of the underlying $c\mathrm{KP}_{1,M}$ integrable hierarchy.

Comparing the corresponding negative-grade symmetry flows in the algebraic and Sato formulations (cf. (117)) and taking into account the explicit form of $H^{(n)}_{\lambda_M}$ (first relation (196)), we find

$$\delta^{(-n)}_{H_{\lambda_M}} = -\partial/\partial \overset{(M+1)}{t}_{-n} + \frac{1}{M+1}\delta^{(-n)}_{\mathcal{A}=\mathbb{1}} \simeq -\partial/\partial \overset{(M+1)}{t}_{-n} \tag{227}$$

since, as explained above, the flows $\delta^{(-n)}_{\mathcal{A}=\mathbb{1}}$ vanish identically. Therefore, to obtain solutions for (224) it is sufficient to consider Darboux–Bäcklund transformations of $c\mathrm{KP}_{1,M}$-based extended KP-type hierarchy of the form (146)–(148) (for $R=1$) where the "light-cone" time derivative $\bar{\partial}$ from (224) is identified, up to an overall sign according to (227), with the lowest additional "isospectral" flow $\bar{\partial} \equiv -\delta^{(-1)}_{H_{\lambda_M}} = \partial/\partial \overset{(M+1)}{t}_{-1}$.

8.1. *Darboux–Bäcklund transformations preserving additional symmetries*

Let us recall that Darboux–Bäcklund (DB) transformations within Sato pseudo-differential approach are defined as "gauge" transformations of special kind on the pertinent Lax operator of the general (unconstrained) KP hierarchy:

$$\mathcal{L} \to \tilde{\mathcal{L}} = T_\phi \mathcal{L} T_\phi^{-1}\,, \qquad T_\phi \equiv \phi D \phi^{-1}\,, \tag{228}$$

which preserve the isospectral (Sato evolution) equations (89):

$$\frac{\partial}{\partial t_n}\tilde{\mathcal{L}} = \left[\frac{\partial}{\partial t_n} T_\phi T_\phi^{-1} + T_\phi \mathcal{L}^n_+ T_\phi^{-1}, \tilde{\mathcal{L}}\right] = [\tilde{\mathcal{L}}^n_+, \tilde{\mathcal{L}}]\,. \tag{229}$$

For the second equality in (229) to be true, the function ϕ must be an eigenfunction (94). Similarly, one can define adjoint-DB transformation:

$$\mathcal{L} \to \widehat{\mathcal{L}} = T_\psi^{*-1} \mathcal{L} T_\psi^{*}\,, \qquad T_\psi \equiv \psi D \psi^{-1}\,, \tag{230}$$

where the function ψ is an adjoint eigenfunction (94). From Eq. (93) one finds the (adjoint) DB transformations of the tau-function:

$$\tau \to \tilde{\tau} = \tau\phi\,, \qquad \tau \to \hat{\tau} = -\tau\psi\,. \tag{231}$$

In the case of constrained $c\mathrm{KP}_{R,M}$ hierarchies (100), the (adjoint) DB transformations (228), (230) must in addition preserve also the constrained form of $\mathcal{L} \equiv \mathcal{L}_{R,M}$. The transformed Lax operator and its inverse are of the form (using the pseudo-differential operator identities (99)):

$$\tilde{\mathcal{L}} \equiv \tilde{\mathcal{L}}_{R,M} = \tilde{\mathcal{L}}_+ + (T_\phi \mathcal{L}(\phi)) D^{-1} \phi^{-1} + \sum_{i=1}^{M} (T_\phi(\Phi_i)) D^{-1} (T_\phi^{*-1}(\Psi_i))\,, \tag{232}$$

$$\tilde{\mathcal{L}}^{-1} = (T_\phi \mathcal{L}^{-1}(\phi)) D^{-1} \phi^{-1} + \sum_{a=1}^{M+R} (T_\phi(\Phi_a^{(-1)})) D^{-1} (T_\phi^{*-1}(\Psi_a^{(-1)}))\,. \tag{233}$$

Similarly for adjoint DB transformations we have:

$$\widehat{\mathcal{L}} \equiv \widehat{\mathcal{L}}_{R,M} = \widehat{\mathcal{L}}_+ - \psi^{-1} D^{-1} (T_\psi \mathcal{L}^*(\psi)) + \sum_{i=1}^{M} (T_\psi^{*-1}(\Phi_i)) D^{-1} (T_\psi(\Psi_i))\,, \tag{234}$$

$$\widehat{\mathcal{L}}^{-1} = -\psi^{-1} D^{-1} (T_\psi \mathcal{L}^{*-1}(\psi)) + \sum_{a=1}^{M+R} (T_\psi^{*-1}(\Phi_a^{(-1)})) D^{-1} (T_\psi(\Psi_a^{(-1)}))\,, \tag{235}$$

where the shorthand notations (116) have been used. Since for generic (adjoint) eigenfunctions ϕ (ψ) the negative pseudo-differential part of the transformed Lax operator (232)–(234) has one more term than the original $\mathcal{L} \equiv \mathcal{L}_{R,M}$, there are two types of conditions implying the vanishing of one superfluous term on the r.h.s. of Eqs. (232)–(234), so that the constrained form of $\mathcal{L} \equiv \mathcal{L}_{R,M}$ is preserved:

$$\text{either} \quad \mathcal{L}(\phi) = 0\,, \quad \text{or} \quad \phi = \Phi_{i_0} \to T_\phi(\Phi_{i_0}) = 0 \tag{236}$$

for some fixed index i_0 between 1 and M, and for adjoint-DB transformations:

$$\text{either} \quad \mathcal{L}^*(\psi) = 0\,, \quad \text{or} \quad \psi = \Psi_{j_0} \to T_\psi(\Psi_{j_0}) = 0 \tag{237}$$

for some fixed index j_0 between 1 and M. Due to relations (108) the first type of conditions (236)–(237) are satisfied by the (adjoint) eigenfunctions entering inverse powers of the Lax operator:

$$\phi = L_M(\bar{\varphi}_{a_0})\,, \qquad \psi = \bar{\psi}_{b_0} \tag{238}$$

for some fixed indices a_0, b_0 between 2 and $M+R$.

Here we are interested in (adjoint) DB transformations which, in addition to preserving the constrained form of $c\mathrm{KP}_{R,M}$ Lax operators, preserve also their additional symmetries. The general case is discussed in the appendix. Here we will concentrate on (adjoint) DB transformations which preserve the extended set of "isospectral" flow equations of $c\mathrm{KP}_{R,M}$-based extended KP-type hierarchies (146)–(148). For the (adjoint) DB-transformed flow-generating operators $\mathcal{M}^{(-n)}_{E_k}$ we get

$$\begin{aligned}\delta^{(-n)}_{E_k} T_\phi T_\phi^{-1} + T_\phi \mathcal{M}^{(-n)}_{E_k} T_\phi^{-1} &= (T_\phi(\mathcal{M}^{(-n)}_{E_k}(\phi) - \delta^{(-n)}_{E_k}\phi))D^{-1}\phi^{-1} \\ &\quad + \sum_{s=1}^{n} T_\phi(\Phi_k^{(-n-1+s)})D^{-1}T_\phi^{*-1}(\Psi_k^{(-s)})\,,\end{aligned} \tag{239}$$

$$\begin{aligned}(\delta^{(-n)}_{E_k} T_\psi^{*-1})T_\psi^* + T_\psi^{*-1}\mathcal{M}^{(-n)}_{E_k} T_\psi^* &= -\psi^{-1}D^{-1}(T_\psi((\mathcal{M}^{(-n)}_{E_k})^*(\psi) + \delta^{(-n)}_{E_k}\psi)) \\ &\quad + \sum_{s=1}^{n} T_\psi^{*-1}(\Phi_k^{(-n-1+s)})D^{-1}T_\psi(\Psi_k^{(-s)})\,.\end{aligned} \tag{240}$$

Here again we have one more term in the transformed operators $\mathcal{M}^{(-n)}_{E_k}$ and again we obtain two types of conditions for vanishing of one superfluous term:

$$\text{either} \quad \delta^{(-n)}_{E_k}\phi = \mathcal{M}^{(-n)}_{E_k}(\phi)\,, \quad \text{or} \quad \phi = \Phi^{(-1)}_{a_0} \equiv L_M(\bar{\varphi}_{a_0}) \to T_\phi(\Phi^{(-1)}_{a_0}) = 0 \tag{241}$$

for some fixed index a_0 between 2 and $M+R$, and for adjoint-DB transformations:

$$\text{either} \quad \delta^{(-n)}_{E_k}\psi = -(\mathcal{M}^{(-n)}_{E_k})^*(\psi)\,, \quad \text{or} \quad \psi = \Psi^{(-1)}_{b_0} \equiv \bar{\psi}_{b_0} \to T_\psi(\Psi^{(-1)}_{b_0}) = 0 \tag{242}$$

for some fixed index b_0 between 2 and $M+R$. The first type of conditions in (241)–(242) are satisfied, due to Eqs. (118), by

$$\phi = \Phi_{i_0}\,, \qquad \psi = \Psi_{j_0} \tag{243}$$

for some fixed indices i_0, j_0 between 1 and M.

Relations (236)–(238) and (241)–(243) define the set of allowed (adjoint) DB transformations (228) and (230), which preserve the form of the $c\mathrm{KP}_{R,M}$-based extended integrable hierarchies (146)–(148). From (232)–(235) we find the (adjoint) DB-transformations of the corresponding (adjoint) eigenfunctions — building blocks of $\mathcal{L}$ (100) and $\mathcal{M}^{(-n)}_{E_k}$ (145):

- For the first choice of DB-generating (adjoint) eigenfunctions (243) we have:

$$\tilde{\Phi}^{(n)}_{i_0} = T_\phi(\Phi^{(n+1)}_{i_0}) \,, \quad \tilde{\Psi}^{(n)}_{i_0} = (T_\phi)^{*-1}(\Psi^{(n-1)}_{i_0}) \quad \text{for } n \geq 2 \,, \qquad \text{with} \quad \tilde{\Psi}_{i_0} = \frac{1}{\phi} \equiv \frac{1}{\Phi_{i_0}} \,, \tag{244}$$

$$\tilde{\Phi}^{(n)}_{i} = T_\phi(\Phi^{(n)}_{i}) \,, \quad \tilde{\Psi}^{(n)}_{i} = T^{*-1}_\phi(\Psi^{(n)}_{i}) \quad \text{for } i \neq i_0 \,, \tag{245}$$

$$\tilde{\Phi}^{(-n)}_{a} = T_\phi(\Phi^{(-n)}_{a}) \,, \quad \tilde{\Psi}^{(-n)}_{a} = T^{*-1}_\phi(\Psi^{(-n)}_{a}) \,, \tag{246}$$

$$\hat{\Phi}_{j_0} = -\frac{1}{\psi} \equiv -\frac{1}{\Psi_{j_0}} \,, \quad \hat{\Phi}^{(n)}_{j_0} = -T^{*-1}_\psi(\Phi^{(n-1)}_{j_0}) \quad \text{for } n \geq 2 \,, \qquad \text{with} \quad \hat{\Psi}^{(n)}_{j_0} = -T_\psi(\Psi^{(n+1)}_{j_0}) \,, \tag{247}$$

$$\hat{\Phi}^{(n)}_{i} = -T^{*-1}_\psi(\Phi^{(n)}_{i}) \,, \quad \hat{\Psi}^{(n)}_{i} = -T_\psi(\Psi^{(n)}_{i}) \,, \quad i \neq j_0 \,, \tag{248}$$

$$\hat{\Phi}^{(-n)}_{a} = -T^{*-1}_\psi(\Phi^{(-n)}_{a}) \,, \quad \hat{\Psi}^{(-n)}_{a} = -T_\psi(\Psi^{(-n)}_{a}) \,. \tag{249}$$

- For the second choice of DB-generating (adjoint) eigenfunctions (238) we obtain:

$$\tilde{\Phi}^{(n)}_{i} = T_\phi(\Phi^{(n)}_{i}) \,, \quad \tilde{\Psi}^{(n)}_{i} = T^{*-1}_\phi(\Psi^{(n)}_{i}) \,, \quad \phi \equiv L_M(\bar{\varphi}_{a_0}) \equiv \Phi^{(-1)}_{a_0} \,, \tag{250}$$

$$\tilde{\Phi}^{(-n)}_{a_0} = T_\phi(\Phi^{(-n-1)}_{a_0}) \,, \quad \tilde{\Psi}^{(-n)}_{a_0} = (T_\phi)^{*-1}(\Psi^{(-n+1)}_{a_0}) \quad \text{for } n \geq 2 \,, \qquad \text{with} \quad \tilde{\Psi}^{(-1)}_{a_0} = \frac{1}{\phi} \equiv \frac{1}{\Phi^{(-1)}_{a_0}} \,, \tag{251}$$

$$\tilde{\Phi}^{(-n)}_{a} = T_\phi(\Phi^{(-n)}_{a}) \,, \quad \tilde{\Psi}^{(-n)}_{a} = T^{*-1}_\phi(\Psi^{(-n)}_{a}) \quad \text{for } a \neq a_0 \,, \tag{252}$$

$$\hat{\Phi}^{(n)}_{i} = -T^{*-1}_\psi(\Phi^{(n)}_{i}) \,, \quad \hat{\Psi}^{(n)}_{i} = -T_\psi(\Psi^{(n)}_{i}) \,, \quad \psi \equiv \bar{\psi}_{b_0} \equiv \Psi^{(-1)}_{b_0} \,, \tag{253}$$

$$\hat{\Phi}^{(-1)}_{b_0} = -\frac{1}{\psi} \equiv -\frac{1}{\Psi^{(-1)}_{b_0}} \,, \quad \hat{\Phi}^{(-n)}_{b_0} = -T^{*-1}_\psi(\Phi^{(-n+1)}_{b_0}) \quad \text{for } n \geq 2 \,, \qquad \text{with} \quad \hat{\Psi}^{(-n)}_{b_0} = -T_\psi(\Psi^{(-n-1)}_{b_0}) \,, \tag{254}$$

$$\hat{\Phi}^{(-n)}_{a} = -T^{*-1}_\psi(\Phi^{(-n)}_{a}) \,, \quad \hat{\Psi}^{(-n)}_{a} = -T_\psi(\Psi^{(-n)}_{a}) \quad \text{for } a \neq b_0 \,. \tag{255}$$

8.2. Iterations of Darboux–Bäcklund transformations

The general Darboux–Bäcklund orbit consists of successive applications of the allowed (adjoint) DB transformations (244)–(255). During iteration of these DB transformations the following generalization of Wronskian determinants appears:

$$\tilde{W}_{m;l}[\phi_1,\ldots,\phi_m;\psi_1,\ldots,\psi_l] = \det\left\| \begin{matrix} \phi_1 & \cdots & \phi_m \\ \partial\phi_1 & \cdots & \partial\phi_m \\ \vdots & \ddots & \vdots \\ \partial^{m-l-1}\phi_1 & \cdots & \partial^{m-l-1}\phi_m \\ \partial^{-1}(\phi_1\psi_1) & \cdots & \partial^{-1}(\phi_m\psi_1) \\ \vdots & \ddots & \vdots \\ \partial^{-1}(\phi_1\psi_l) & \cdots & \partial^{-1}(\phi_m\psi_l) \end{matrix} \right\| , \quad m \geq l , \tag{256}$$

the ordinary Wronskians being $W_m[\phi_1,\ldots,\phi_m] \equiv \det\|\partial^{\alpha-1}\phi_\beta\|_{\alpha,\beta=1,\ldots,m}$.

Let us recall that iterations of DB transformations with T_ϕ on any function f are given by

$$T^{(k-1;0)}_{\phi_k}\cdots T^{(0;0)}_{\phi_1}(f) = \frac{W_{k+1}[\phi_1,\ldots,\phi_k,f]}{W_k[\phi_1,\ldots,\phi_k]} , \tag{257}$$

where

$$T^{(l-1;0)}_{\phi_l} \equiv \phi^{(l-1;0)}_l D(\phi^{(l-1;0)}_l)^{-1} , \qquad \phi^{(l-1;0)}_l = T^{(l-2;0)}_{\phi_{l-1}}\cdots T^{(0;0)}_{\phi_1}(\phi_l) . \tag{258}$$

Equation (257) follows from the identity (the Jacobi expansion theorem for Wronskians):

$$\begin{aligned} &W_k[\phi_1,\ldots,\phi_{k-1},f]\overleftrightarrow{\partial}W_k[\phi_1,\ldots,\phi_{k-1},g] \\ &\qquad = W_{k-1}[\phi_1,\ldots,\phi_{k-1}]W_{k+1}[\phi_1,\ldots,\phi_{k-1},f,g] , \end{aligned} \tag{259}$$

where f,g are arbitrary functions.

Iterations of adjoint DB transformations (defined with T^{*-1}_ϕ) on any function f read:

$$(T^{(k-1;0)}_{\phi_k})^{*-1}\cdots(T^{(0;0)}_{\phi_1})^{*-1}(f) = -\frac{\tilde{W}_{k;1}[\phi_1,\ldots,\phi_k;f]}{W_k[\phi_1,\ldots,\phi_k]} . \tag{260}$$

Equation (260) follows from an identity generalizing (259) and involving generalized Wronskian-like determinants (256):

$$\begin{aligned} &W_k[\phi_1,\ldots,\phi_k]\overleftrightarrow{\partial}\tilde{W}_{k+1;1}[\phi_1,\ldots,\phi_k,f;g] \\ &\qquad = -W_{k+1}[\phi_1,\ldots,\phi_k,f]\tilde{W}_{k;1}[\phi_1,\ldots,\phi_k;g] . \end{aligned} \tag{261}$$

Both Wronskian(-like) identities (259) and (261) can be further generalized to

$$\tilde{W}_{m;l+1}[\phi_1,\ldots,\phi_m;\psi_1,\ldots,\psi_l,f] \overset{\leftrightarrow}{\partial} \tilde{W}_{m;l+1}[\phi_1,\ldots,\phi_m;\psi_1,\ldots,\psi_l,g]$$
$$= \tilde{W}_{m;l}[\phi_1,\ldots,\phi_m;\psi_1,\ldots,\psi_l]\tilde{W}_{m;l+2}[\phi_1,\ldots,\phi_m;\psi_1,\ldots,\psi_l,f,g]\,, \tag{262}$$

$$\tilde{W}_{m;n}[\phi_1,\ldots,\phi_m;\psi_1,\ldots,\psi_n] \overset{\leftrightarrow}{\partial} \tilde{W}_{m+1;n+1}[\phi_1,\ldots,\phi_m,f;\psi_1,\ldots,\psi_n,g]$$
$$= -\tilde{W}_{m+1;n}[\phi_1,\ldots,\phi_m,f;\psi_1,\ldots,\psi_n]\tilde{W}_{m;n+1}[\phi_1,\ldots,\phi_m;\psi_1,\ldots,\psi_n,g]\,. \tag{263}$$

Using (262)–(263) we are able to find the explicit Wronskian-like expression for arbitrary mixed iterations of DB and adjoint-DB transformations of the type (244)–(255):

$$(T^{(m;n-1)}_{\psi_n})^{*-1}\cdots(T^{(m;0)}_{\psi_1})^{*-1}T^{(m-1;0)}_{\phi_m}\cdots T^{(0;0)}_{\phi_1}(f)$$
$$= \frac{\tilde{W}_{m+1;n}[\phi_1,\ldots,\phi_m,f;\psi_1,\ldots,\psi_n]}{\tilde{W}_{m;n}[\phi_1,\ldots,\phi_m;\psi_1,\ldots,\psi_n]}\,,\quad \text{for } m\geq n\,, \tag{264}$$

$$T^{(m;n-1)}_{\psi_n}\cdots T^{(m;0)}_{\psi_1}(T^{(m-1;0)}_{\phi_m})^{*-1}\cdots(T^{(0;0)}_{\phi_1})^{*-1}(f)$$
$$= -\frac{\tilde{W}_{m;n+1}[\phi_1,\ldots,\phi_m;\psi_1,\ldots,\psi_n,f]}{\tilde{W}_{m;n}[\phi_1,\ldots,\phi_m;\psi_1,\ldots,\psi_n]}\,,\quad \text{for } m\geq n+1\,, \tag{265}$$

where $T^{(l-1;0)}_{\phi_l}$ is the same as in (258), and where

$$\begin{aligned}(T^{(m;l-1)}_{\psi_l})^{*-1} &\equiv -(\psi_l^{(m;l-1)})^{-1}D^{-1}\psi_l^{(m;l-1)}\,,\\ \psi_l^{(m;l-1)} &= T^{(m;l-2)}_{\psi_{l-1}}\cdots T^{(m;0)}_{\psi_1}(T^{(m-1;0)}_{\phi_m})^{*-1}\cdots(T^{(0;0)}_{\phi_1})^{*-1}(\psi_l)\,.\end{aligned} \tag{266}$$

Similarly we have

$$(T^{(m-1;n)}_{\phi_m})^{*-1}\cdots(T^{(0;n)}_{\phi_1})^{*-1}T^{(0;n-1)}_{\psi_n}\cdots T^{(0;0)}_{\psi_1}(f)$$
$$= \frac{\tilde{W}_{n+1;m}[\psi_1,\ldots,\psi_n,f;\phi_1,\ldots,\phi_m]}{\tilde{W}_{n;m}[\psi_1,\ldots,\psi_n;\phi_1,\ldots,\phi_m]}\,,\quad \text{for } n\geq m\,, \tag{267}$$

$$T^{(m-1;n)}_{\phi_m}\cdots T^{(0;n)}_{\phi_1}(T^{(0;n-1)}_{\psi_n})^{*-1}\cdots(T^{(0;0)}_{\psi_1})^{*-1}(f)$$
$$= -\frac{\tilde{W}_{n;m+1}[\psi_1,\ldots,\psi_n;\phi_1,\ldots,\phi_m,f]}{\tilde{W}_{n;m}[\psi_1,\ldots,\psi_n;\phi_1,\ldots,\phi_m]}\,,\quad \text{for } m\geq n+1\,, \tag{268}$$

where

$$\begin{aligned}T^{(0;l-1)}_{\psi_l} &\equiv \psi_l^{(0;l-1)}D(\psi_l^{(0;l-1)})^{-1}\,,\\ \psi_l^{(0;l-1)} &= T^{(0;l-2)}_{\psi_{l-1}}\cdots T^{(0;0)}_{\psi_1}(\psi_l)\,,\end{aligned} \tag{269}$$

$$\begin{aligned}(T^{(l-1;n)}_{\phi_l})^{*-1} &\equiv -(\phi_l^{(l-1;n)})^{-1}D^{-1}\phi_l^{(l-1;n)}\\ \phi_l^{(l-1;n)} &= T^{(l-2;n)}_{\phi_{l-1}}\cdots T^{(;n0)}_{\phi_1}(T^{(0;n-1)}_{\psi_n})^{*-1}\cdots(T^{(0;0)}_{\psi_1})^{*-1}(\phi_l)\,.\end{aligned} \tag{270}$$

Remark. Before proceeding lets us recall the following simple property. A pair of successive DB and adjoint-DB transformations of the first type (244)–(249) w.r.t. $\phi \equiv \Phi_{i_0}$ and $\psi \equiv \Psi_{i_0}$, respectively, yield an identity combined transformation due to (244). Similar property applies also to the pair of successive DB and adjoint-DB transformations of the second type (250)–(255) w.r.t. $\phi \equiv \Phi^{-1}_{a_0}$ and $\psi \equiv \Psi^{-1}_{a_0}$, respectively, due to (251).

Therefore, the general DB orbit can be labeled by two nonnegative integer-valued vectors:

$$\mathbf{m} \equiv (m_1, \ldots, m_M, \bar{m}_2, \ldots, \bar{m}_{M+R}) , \tag{271}$$

$$\mathbf{n} \equiv (n_1, \ldots, n_M, \bar{n}_2, \ldots, \bar{n}_{M+R}) , \tag{272}$$

where each entry m_s in (271) indicates m_s DB steps w.r.t. $\phi = \Phi_{i_s}$, and each entry $\bar{m}_s$ indicates $\bar{m}_s$ DB steps w.r.t. $\phi = \Phi^{(-1)}_{a_s}$. Similarly, each entry n_s in (272) indicates n_s adjoint-DB steps w.r.t. $\psi = \Psi_{i_s}$, and each entry $\bar{n}_s$ indicates $\bar{n}_s$ adjoint-DB steps w.r.t. $\psi = \Psi^{(-1)}_{a_s}$. According to the above remark one of the two integers in each pair (m_s, n_s) and $(\bar{m}_s, \bar{n}_s)$ must be zero.

Taking into account (244)–(255), the (adjoint) DB iterations along the general DB orbit are given (up to overall signs) by (264)–(265) and (267)–(268) where (using the shorthand notations (110), (116)):

$$\begin{aligned}\{\phi_1, \ldots, \phi_m\} &\equiv \{\Phi(\mathbf{m})\} \\ &= \{\Phi^{(1)}_1, \ldots, \Phi^{(m_1)}_1; \ldots; \Phi^{(1)}_M, \ldots, \Phi^{(m_M)}_M; \\ &\qquad \Phi^{(-1)}_2, \ldots, \Phi^{(-\bar{m}_2)}_2; \ldots; \Phi^{(-1)}_{\bar{M}}, \ldots, \Phi^{(-\bar{m}_{\bar{M}})}_{\bar{M}}\} ,\end{aligned} \tag{273}$$

$$\begin{aligned}\{\psi_1, \ldots, \psi_n\} &\equiv \{\Psi(\mathbf{n})\} \\ &= \{\Psi^{(1)}_1, \ldots, \Psi^{(n_1)}_1; \ldots; \Psi^{(1)}_M, \ldots, \Psi^{(n_M)}_M; \\ &\qquad \Psi^{(-1)}_2, \ldots, \Phi^{(-\bar{n}_2)}_2; \ldots; \Psi^{(-1)}_{\bar{M}}, \ldots, \Phi^{(-\bar{n}_{\bar{M}})}_{\bar{M}}\} ,\end{aligned} \tag{274}$$

with

$$|\mathbf{m}| \equiv \sum_{k=1}^{M} m_k + \sum_{l=2}^{M+R} \bar{m}_l , \quad |\mathbf{n}| \equiv \sum_{k=1}^{M} n_k + \sum_{l=2}^{M+R} \bar{n}_l , \quad \bar{M} \equiv M + R . \tag{275}$$

In what follows, (adjoint) DB-transformed tau-function and (adjoint) eigenfunctions of the $c\mathrm{KP}_{R,M}$ hierarchy (100) along the DB orbit (273)–(274) will be denoted as $\tau_{(\mathbf{m};\mathbf{n})}$, $\Phi_{i,(\mathbf{m};\mathbf{n})}$ and $\Psi_{i,(\mathbf{m};\mathbf{n})}$.

Using the shorthand notations from (273)–(274) and the notation for the generalized Wronskian-like determinants (256), the general DB solutions for the tau-function and the constituent (adjoint) eigenfunctions of $c\mathrm{KP}_{R,M}$ hierarchies (100) can be written in the following compact form:

$$\frac{\tau_{(\mathbf{m};\mathbf{n})}}{\tau^{(0;0)}} = (-1)^{|\mathbf{n}|(|\mathbf{n}|-1)/2}\tilde{W}_{\mathbf{m};\mathbf{n}}[\{\Phi(\mathbf{m})\};\{\Psi(\mathbf{n})\}]\,, \quad \text{for } |\mathbf{m}| \geq |\mathbf{n}|\,, \tag{276}$$

$$\Phi_{i,(\mathbf{m};\mathbf{n})} = (-1)^{|\mathbf{n}|}\frac{\tilde{W}_{\mathbf{m}_+^{(i)};\mathbf{n}}[\{\Phi(\mathbf{m}_+^{(i)})\};\{\Psi(\mathbf{n})\}]}{\tilde{W}_{\mathbf{m};\mathbf{n}}[\{\Phi(\mathbf{m})\};\{\Psi(\mathbf{n})\}]}\,, \quad \text{for } |\mathbf{m}| \geq |\mathbf{n}|\,,\ n_i = 0\,, \tag{277}$$

$$\Phi_{i,(\mathbf{m};\mathbf{n})} = (-1)^{|\mathbf{n}|-1}\frac{\tilde{W}_{\mathbf{m};\mathbf{n}_-^{(i)}}[\{\Phi(\mathbf{m})\};\{\Psi(\mathbf{n}_-^{(i)})\}]}{\tilde{W}_{\mathbf{m};\mathbf{n}}[\{\Phi(\mathbf{m})\};\{\Psi(\mathbf{n})\}]}\,, \quad \text{for } |\mathbf{m}| \geq |\mathbf{n}|\,,\ n_i \geq 1\,, \tag{278}$$

$$\Psi_{i,(\mathbf{m};\mathbf{n})} = (-1)^{|\mathbf{n}|-1}\frac{\tilde{W}_{\mathbf{m};\mathbf{n}_+^{(i)}}[\{\Phi(\mathbf{m})\};\{\Psi(\mathbf{n}_+^{(i)})\}]}{\tilde{W}_{\mathbf{m};\mathbf{n}}[\{\Phi(\mathbf{m})\};\{\Psi(\mathbf{n})\}]}\,, \quad \text{for } |\mathbf{m}| \geq |\mathbf{n}|+1\,,\ m_i = 0\,, \tag{279}$$

$$\Psi_{i,(\mathbf{m};\mathbf{n})} = (-1)^{|\mathbf{n}|}\frac{\tilde{W}_{\mathbf{m}_-^{(i)};\mathbf{n}}[\{\Phi(\mathbf{m}_-^{(i)})\};\{\Psi(\mathbf{n})\}]}{\tilde{W}_{\mathbf{m};\mathbf{n}}[\{\Phi(\mathbf{m})\};\{\Psi(\mathbf{n})\}]}\,, \quad \text{for } |\mathbf{m}| \geq |\mathbf{n}|\,,\ m_i \geq 1\,. \tag{280}$$

In Eqs. (277)–(280) the following notations are used: the vectors $\mathbf{m}_\pm^{(i)}$ are obtained from $\mathbf{m}$ (271) by the shift $m_i \to m_i \pm 1$ and, similarly, the vectors $\mathbf{n}_\pm^{(i)}$ are obtained from $\mathbf{n}$ (272) by the shift $n_i \to n_i \pm 1$.

Similarly we obtain

$$\frac{\tau_{(\mathbf{m};\mathbf{n})}}{\tau^{(0;0)}} = (-1)^{|\mathbf{n}|(|\mathbf{n}|-1)/2}\tilde{W}_{\mathbf{n};\mathbf{m}}[\{\Psi(\mathbf{n})\};\{\Phi(\mathbf{m})\}]\,, \quad \text{for } |\mathbf{n}| \geq |\mathbf{m}|\,, \tag{281}$$

$$\Phi_{i,(\mathbf{m};\mathbf{n})} = (-1)^{|\mathbf{n}|}\frac{\tilde{W}_{\mathbf{n};\mathbf{m}_+}[\{\Psi(\mathbf{n})\};\{\Phi(\mathbf{m}_+)\}]}{\tilde{W}_{\mathbf{n};\mathbf{m}}[\{\Psi(\mathbf{n})\};\{\Phi(\mathbf{m})\}]}\,, \quad \text{for } |\mathbf{n}| \geq |\mathbf{m}|+1\,,\ n_i = 0\,, \tag{282}$$

$$\Phi_{i,(\mathbf{m};\mathbf{n})} = (-1)^{|\mathbf{n}|-1}\frac{\tilde{W}_{\mathbf{n}_-^{(i)};\mathbf{m}}[\{\Psi(\mathbf{n}_-^{(i)})\};\{\Phi(\mathbf{m})\}]}{\tilde{W}_{\mathbf{n};\mathbf{m}}[\{\Psi(\mathbf{n})\};\{\Phi(\mathbf{m})\}]}\,, \quad \text{for } |\mathbf{n}| \geq |\mathbf{m}|+1\,,\ n_i \geq 1\,, \tag{283}$$

$$\Psi_{i,(\mathbf{m};\mathbf{n})} = (-1)^{|\mathbf{n}|-1}\frac{\tilde{W}_{\mathbf{n}_+^{(i)};\mathbf{m}}[\{\Psi(\mathbf{n}_+^{(i)})\};\{\Phi(\mathbf{m})\}]}{\tilde{W}_{\mathbf{n};\mathbf{m}}[\{\Psi(\mathbf{n})\};\{\Phi(\mathbf{m})\}]}\,, \quad \text{for } |\mathbf{n}| \geq |\mathbf{m}|\,,\ m_i = 0\,, \tag{284}$$

$$\Psi_{i,(\mathbf{m};\mathbf{n})} = (-1)^{|\mathbf{n}|}\frac{\tilde{W}_{\mathbf{n};\mathbf{m}_-^{(i)}}[\{\Psi(\mathbf{n})\};\{\Phi(\mathbf{m}_-^{(i)})\}]}{\tilde{W}_{\mathbf{n};\mathbf{m}}[\{\Psi(\mathbf{n})\};\{\Phi(\mathbf{m})\}]}\,, \quad \text{for } |\mathbf{n}| \geq |\mathbf{m}|\,,\ m_i \geq 1\,. \tag{285}$$

8.3. *Multiple-Wronskian solutions*

Let us take a closer look at the DB solution for the tau-function (276) of $c\mathrm{KP}_{R,M}$ integrable hierarchies (100) in the case of DB orbits with $\bar{m}_2 = \cdots = \bar{m}_{M+R} = 0$ and $n_1 = \cdots = n_M = 0$ in (273)–(274) (below $\bar{M} \equiv M + R$):

$$\tilde{W}_{\mathbf{m},\mathbf{n}}[\Phi_1^{(1)},\ldots,\Phi_1^{(m_1)},\ldots,\Phi_M^{(1)},\ldots,\Phi_M^{(m_M)};\Psi_2^{(-1)},\ldots,\Psi_2^{(-\bar{n}_2)},\ldots,\ldots,\Psi_{\bar{M}}^{(-\bar{n}_{\bar{M}})}]$$

$$= \det \begin{Vmatrix} \Phi_1^{(1)} & \cdots & \Phi_1^{(m_1)} & \cdots & \cdots & \Phi_M^{(m_M)} \\ \vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\ \partial^{m-n-1}\Phi_1^{(1)} & \cdots & \partial^{m-n-1}\Phi_1^{(m_1)} & \cdots & \cdots & \partial^{m-n-1}\Phi_M^{(m_M)} \\ \partial^{-1}(\Phi_1^{(1)}\Psi_2^{(-1)}) & \cdots & \partial^{-1}(\Phi_1^{(m_1)}\Psi_2^{(-1)}) & \cdots & \cdots & \partial^{-1}(\Phi_M^{(m_M)}\Psi_2^{(-1)}) \\ \vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\ \partial^{-1}(\Phi_1^{(1)}\Psi_2^{(-\bar{n}_2)}) & \cdots & \partial^{-1}(\Phi_1^{(m_1)}\Psi_2^{(-\bar{n}_2)}) & \cdots & \cdots & \partial^{-1}(\Phi_M^{(m_M)}\Psi_2^{(-\bar{n}_2)}) \\ \vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\ \partial^{-1}(\Phi_1^{(1)}\Psi_{\bar{M}}^{(-1)}) & \cdots & \partial^{-1}(\Phi_1^{(m_1)}\Psi_{\bar{M}}^{(-1)}) & \cdots & \cdots & \partial^{-1}(\Phi_M^{(m_M)}\Psi_{\bar{M}}^{(-1)}) \\ \vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\ \partial^{-1}(\Phi_1^{(1)}\Psi_{\bar{M}}^{(-\bar{n}_{\bar{M}})}) & \cdots & \partial^{-1}(\Phi_1^{(m_1)}\Psi_{\bar{M}}^{(-\bar{n}_{\bar{M}})}) & \cdots & \cdots & \partial^{-1}(\Phi_M^{(m_M)}\Psi_{\bar{M}}^{(-\bar{n}_{\bar{M}})}) \end{Vmatrix} . \tag{286}$$

In Refs. 42 the following properties were shown to hold for SEP (square eigenfunction potential, cf. (96)–(98)) functions of the type entering in (286), namely, such that the second adjoint eigenfunction belongs to the set of additional "isospectral" flow generating (adjoint) eigenfunctions (145), (148) whereas the first eigenfunction transforms homogeneously w.r.t. the corresponding additional "isospectral" flows as in (147). SEP functions of the form (using shorthand notations (110), (116)):

$$\overset{(k)}{F}_{i,N} \equiv \partial^{-1}(\Phi_i^{(N)}\Psi_k^{(-1)}) \quad N \geq 1\,,\ i = 1,\ldots,M\,,\ k = 2,\ldots,M+R\,, \tag{287}$$

which enter (286), obey eigenfunction-type purely differential equations (analogous to (94)) w.r.t. the additional "isospectral" parameters $\{\overset{(k)}{t}_{-n}\}_{n=1}^{\infty}$:

$$\partial/\partial\overset{(k)}{t}_{-n}\overset{(k)}{F}_{i,N} = \left[\overset{(k)}{\partial}{}^{n} + \cdots\right]\overset{(k)}{F}_{i,N}\,, \qquad \overset{(k)}{\partial} \equiv \partial/\partial\overset{(k)}{t}_{-1}\,, \tag{288}$$

where the dots indicate lower-order derivative terms w.r.t. $\overset{(k)}{\partial}$ with coefficients depending on the additional "isospectral" flow generating (adjoint) eigenfunctions (145), (148). Furthermore, for the rest of the SEP's in (286) we have

$$\partial^{-1}(\Phi_i^{(N)}\Psi_k^{(-\bar{n})}) = \overset{(k)}{\partial}{}^{\bar{n}-1}\overset{(k)}{F}_{i,N} + \cdots\,, \qquad \bar{n} \geq 2\,, \tag{289}$$

where the dots indicate terms which yield vanishing contribution in the Wronskian-like determinant (286). Thereby (286) acquires the following *multiple-Wronskian* form:

$$\mathcal{W}_{\mathbf{m},\mathbf{n}}[\Phi_1^{(1)},\ldots,\Phi_M^{(m_M)},\overset{(2)}{F}_{1,1},\ldots,\overset{(2)}{F}_{M,m_M},\ldots,\overset{(\bar M)}{F}_{1,1},\ldots,\overset{(\bar M)}{F}_{M,m_M}]$$

$$= \det \left\| \begin{matrix} \Phi_1^{(1)} & \cdots & \Phi_1^{(m_1)} & \cdots & \cdots & \Phi_M^{(m_M)} \\ \vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\ \partial^{m-n-1}\Phi_1^{(1)} & \cdots & \partial^{m-n-1}\Phi_1^{(m_1)} & \cdots & \cdots & \partial^{m-n-1}\Phi_M^{(m_M)} \\ \overset{(2)}{F}_{1,1} & \cdots & \overset{(2)}{F}_{1,m_1} & \cdots & \cdots & \overset{(2)}{F}_{M,m_M} \\ \vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\ \overset{(2)}{\partial}{}^{\bar n_2-1}\overset{(2)}{F}_{1,1} & \cdots & \overset{(2)}{\partial}{}^{\bar n_2-1}\overset{(2)}{F}_{1,m_1} & \cdots & \cdots & \overset{(2)}{\partial}{}^{\bar n_2-1}\overset{(2)}{F}_{M,m_M} \\ \vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\ \overset{(\bar M)}{F}_{1,1} & \cdots & \overset{(\bar M)}{F}_{1,m_1} & \cdots & \cdots & \overset{(\bar M)}{F}_{M,m_M} \\ \vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\ \overset{(\bar M)}{\partial}{}^{\bar n_{\bar M}-1}\overset{(\bar M)}{F}_{1,1} & \cdots & \overset{(\bar M)}{\partial}{}^{\bar n_{\bar M}-1}\overset{(\bar M)}{F}_{1,m_1} & \cdots & \cdots & \overset{(\bar M)}{\partial}{}^{\bar n_{\bar M}-1}\overset{(\bar M)}{F}_{M,m_M} \end{matrix} \right\| . \tag{290}$$

where

$$\bar M \equiv M+R\,, \quad \mathbf{m}=(m_1,\ldots,m_M)\,, \quad \mathbf{n}=(\bar n_2,\ldots,\bar n_{\bar M})\,,$$
$$m=\sum_{i=1}^{M} m_i\,, \quad n=\sum_{a=2}^{\bar M}\bar n_a\,, \tag{291}$$

and notations from (288) are used. Accordingly, the expressions (277) and (280) for the DB-transformed (adjoint) eigenfunctions can be written, using notations in (290)–(291), as a ratio of multiple Wronskians (290):

$$\Phi_{i,(\mathbf{m};\mathbf{n})} = (-1)^n \frac{\mathcal{W}_{\mathbf{m}_+^{(i)};\mathbf{n}}[\ldots,\Phi_i^{(1)},\ldots,\Phi_i^{(m_i)},\Phi_i^{(m_i+1)},\ldots]}{\mathcal{W}_{\mathbf{m};\mathbf{n}}[\ldots,\Phi_i^{(1)},\ldots,\Phi_i^{(m_i)},\ldots]}\,, \quad \text{for } m\ge n\,, \tag{292}$$

$$\Psi_{i,(\mathbf{m};\mathbf{n})} = (-1)^n \frac{\mathcal{W}_{\mathbf{m}_-^{(i)};\mathbf{n}}[\ldots,\Phi_i^{(1)},\ldots,\Phi_i^{(m_i-1)},\ldots]}{\mathcal{W}_{\mathbf{m};\mathbf{n}}[\ldots,\Phi_i^{(1)},\ldots,\Phi_i^{(m_i-1)},\Phi_i^{(m_i)},\ldots]}\,, \quad \text{for } m\ge n\,,\ m_i\ge 1\,. \tag{293}$$

In particular, we are interested in DB orbits passing through the "free" initial point, i.e. where the initial hierarchy is given by the "free" Lax operator $\mathcal{L}=D^R$. In this case $\Phi_i^{(N)}$ and $\Psi_i^{(N)}$ are "free" (adjoint) eigenfunctions given by

$$\Phi_i^{(N)}(t)=\int d\lambda\,\lambda^{NR}\varphi_i(\lambda)e^{\xi(t,\lambda)}\,, \quad \Psi_i^{(N)}(t)=\int d\lambda\,\lambda^{NR}\psi_i(\lambda)e^{-\xi(t,\lambda)}\,, \tag{294}$$

with the same notations as in (92) and (95), where $\varphi_i(\lambda)$ and $\psi_i(\lambda)$ are arbitrary spectral densities. Accordingly, the SEP functions, which appear as matrix elements of the generalized Wronskian-like determinants (256) entering the DB solutions (276)–(285), acquire the following "free" form (i.e set the tau-function $\tau(t) = 1$ in Eq. (97)):

$$\partial^{-1}(\Phi_i^{(N_1)}(t)\Psi_j^{(N_2)}(t)) = \iint d\lambda\, d\mu\, \lambda^{N_1 R}\mu^{N_2 R}\frac{\varphi_i(\lambda)\psi_j(\mu)}{\lambda-\mu}e^{\xi(t,\lambda)-\xi(t,\mu)}\,. \qquad (295)$$

Similarly, in the "free" case the functions $\overset{(k)}{F}_{i,N}$ satisfy the linearized Eq. (288) and, therefore, are given by expressions similar to (294):

$$\overset{(k)}{F}_{i,N}(\overset{(k)}{t}) = \int d\mu\, \overset{(k)}{f}_{i,N}(\mu)e^{\xi(\overset{(k)}{t},\mu)}\,, \qquad \xi(\overset{(k)}{t},\mu) = \sum_{n=1}^{\infty}\mu^n \overset{(k)}{t}_n \qquad (296)$$

with arbitrary spectral densities $\overset{(k)}{f}_{i,N}(\mu)$.

Substituting (294)–(296) into (292)–(293), or in the more general DB-orbit expressions (276)–(285), we obtain explicit DB solutions in generalized Wronskian-like form (256), in particular, in multiple-Wronskian (290) form of the whole class of $c\mathrm{KP}_{R,M}$ reduced KP hierarchies (100).

In the special case $R = 1$, plugging into $A^{(0)}$ (225) the expressions for $\Phi_1,\ldots,\Phi_M$ and $\Psi_1,\ldots,\Psi_M$ given by (292)–(293) or more generally by (276)–(285) with the underlying (adjoint) eigenfunctions and SEP functions of the simple "free" forms (294)–(296), we obtain explicit DB solutions for $\mathrm{SL}(M+1)/\mathrm{U}(1)\times\,\mathrm{SL}(M)$ gauged WZNW field equations of motion (224). These solutions depend, apart from the original "light-cone" coordinates $(x \equiv x_-, x_+)$, identified within the context of the underlying extended $c\mathrm{KP}_{1,M}$ hierarchy with the lowest "isospectral" flow parameters $x \equiv x_- \equiv t_1$ and $x_+ \equiv \overset{(M+1)}{t}_{-1}$, also on the rest of the infinite set of "isospectral" times:

$$\Big(t_1 \equiv x \equiv x_-, t_2, t_3, \ldots;\ \overset{(2)}{t}_1,\ \overset{(2)}{t}_2,\ \overset{(2)}{t}_3, \ldots; \ldots;\ \overset{(M+1)}{t}_{-1} \equiv x_+,\ \overset{(M+1)}{t}_{-2},\ \overset{(M+1)}{t}_{-3}, \ldots\Big)\,. \qquad (297)$$

This additional dependence can be viewed as Abelian symmetry deformations for the corresponding solutions of gauged WZNW equations of motion. Moreover, from the results in the appendix we deduce that the whole non-Abelian additional symmetry algebra of the pertinent $c\mathrm{KP}_{1,M}$ hierarchy ((125) with $R = 1$) acts as symmetry deformation algebra on the space of Darboux–Bäcklund solutions of gauged $\mathrm{SL}(M+1)/\mathrm{U}(1)\times\,\mathrm{SL}(M)$ WZNW equations of motion.

Concluding this section, let us consider as the simplest nontrivial example the $c\mathrm{KP}_{1,1}$-based extended KP-type hierarchy with Lax operator $\mathcal{L} = D + \Phi D^{-1}\Psi$ which is equivalently described within the generalized Drinfeld–Sokolov framework by (206)–(207). In this case the multiple Wronskian tau-function (290) simplifies to the following double-Wronskian form:

$$\mathcal{W}_{m,n} = \det \left\| \begin{matrix} \Phi^{(1)} & \cdots & \Phi^{(m)} \\ \vdots & \ddots & \vdots \\ \partial^{m-n-1}\Phi^{(1)} & \cdots & \partial^{m-n-1}\Phi^{(m)} \\ F_1 & \cdots & F_m \\ \vdots & \ddots & \vdots \\ \bar{\partial}^{n-1}F_1 & \cdots & \bar{\partial}^{n-1}F_m \end{matrix} \right\| , \tag{298}$$

where

$$\Phi^{(l)} \equiv \Phi_1^{(l)}\,, \quad F_l \equiv \overset{(2)}{F_l}\,, \quad m \equiv m_1\,, \quad n \equiv \bar{n}_2\,, \quad \bar{t}_l \equiv \overset{(2)}{t_l}\,, \quad \bar{\partial} \equiv \overset{(2)}{\partial}\,, \tag{299}$$

$$\Phi^{(l)}(t) = \int d\lambda\, \lambda^l \varphi(\lambda) e^{\xi(t,\lambda)}\,, \quad F_l(\bar{t}) = \int d\mu\, f_l(\mu) e^{\xi(t,\mu)}\,. \tag{300}$$

For a special delta-function choice of the spectral densities of $\Phi^{(l)}(t)$ and $F_l(\bar{t})$ (300), the double Wronskian tau-function (298) coincides with the double Wronskian tau-function of the first Ref. 23 which describes the well-known (multi-)dromion solutions[22] of Davey–Stewartson system (159)–(161).

9. Conclusions and Outlook

In the first part of the present paper we discussed the subject of gauging of geometric actions on coadjoint orbits of arbitrary (infinite-dimensional) groups with central extensions (with examples given in Sec. 3). The main tool for our construction of gauged geometric actions on general coadjoint orbits was the group composition law (27) which generalizes the well-known Polyakov–Wiegmann composition formula for ordinary WZNW geometric actions. Furthermore, we have shown that the equations of motions of the pertinent gauged geometric actions possess a "zero-curvature" representation (Sec. 5) on the underlying group coadjoint orbit.

It is a very interesting problem for further study to work out explicit examples of physically relevant gauged geometric actions on coadjoint orbits of infinite-dimensional groups with central extensions (e.g. for those in the examples in Sec. 3) which go beyond the well-known case of Kac–Moody groups yielding the ordinary gauged WZNW actions. For instance, one can study gauging of $\mathbf{W}_\infty$-geometric action (45) with a gauging subgroup whose algebra is the Cartan subalgebra of $\mathbf{W}_\infty$-algebra consisting of differential operators of the type $\{x^n D^n\}_{n=0}^{\infty}$ (cf. Ref. 13).

In the second part of the paper we showed that gauged G/H WZNW field equations can be identified with the lowest negative-grade additional symmetry flow equations of generalized Drinfeld–Sokolov integrable hierarchies based on $\widehat{\mathcal{G}}$ — the loop algebra corresponding to $\mathcal{G}$, the Lie algebra of G. Next we discussed in more detail the case of generalized Drinfeld–Sokolov hierarchies based on $\widehat{\mathrm{SL}}(M+R)$ and explicitly established their equivalence with the class $c\mathrm{KP}_{R,M}$ of constrained (reduced) KP hierarchies. We described in detail the whole loop algebra of additional nonisospectral symmetries of $c\mathrm{KP}_{R,M}$. Adding the flows from the Cartan subalgebra of the underlying additional-symmetry loop algebra, we constructed extended $c\mathrm{KP}_{R,M}$-based integrable hierarchies (Subsec. 6.4) with multiple sets of "isospectral" flows equivalent to certain reductions of multicomponent (matrix) KP hierarchies. Apart from the gauged WZNW field equations mentioned above, other higher-dimensional nonlinear systems such as Davey–Stewartson and N-wave resonant systems were shown to arise as symmetry flow equations of $c\mathrm{KP}_{R,M}$ integrable hierarchies.

Finally, we provided in Sec. 8 a detailed derivation of the general Darboux–Bäcklund solutions for $c\mathrm{KP}_{R,M}$ hierarchies preserving the whole loop algebra of their additional nonisospectral symmetries. These DB solutions involve generalized Wronskian-like determinants, in particular, multiple Wronskians and for $R = 1$ they contain among themselves solutions to the gauged $\mathrm{SL}(M+1)/\mathrm{U}(1)\times\,\mathrm{SL}(M)$ WZNW field equations.

Another important task for further investigation is to study the physical properties of the very broad class of the multiple-Wronskian solutions of $c\mathrm{KP}_{R,M}$ integrable hierarchies given in Sec. 8, which contain as simplest examples the well-known (multi-)dromion solutions.

Also, it is a very interesting problem to exhibit the explicit relation between the present multiple-Wronskian solutions of gauged WZNW models and the recently found[55] solitonic solutions of singular non-Abelian affine Toda field theories.

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Appendix

The (adjoint) DB-transformed additional-symmetry (125) generating operators (111) are of the form:

$$\delta_A^{(n)} T_\phi T_\phi^{-1} + T_\phi \mathcal{M}_A^{(n)} T_\phi^{-1} = (T_\phi(\mathcal{M}_A^{(n)}(\phi) - \delta_A^{(n)}\phi))D^{-1}\phi^{-1} + \sum_{i,j=1}^{M} A_{ij}^{(n)} \sum_{s=1}^{n} T_\phi(\Phi_j^{(n+1-s)})D^{-1}T_\phi^{*-1}(\Psi_i^{(s)})\,, \tag{A.1}$$

$$(\delta_A^{(n)} T_\psi^{*-1})T_\psi^* + T_\psi^{*-1}\mathcal{M}_A^{(n)} T_\psi^* = -\psi^{-1}D^{-1}(T_\psi((\mathcal{M}_A^{(n)})^*(\psi) + \delta_A^{(n)}\psi)) + \sum_{i,j=1}^{M} A_{ij}^{(n)} \sum_{s=1}^{n} T_\psi^{*-1}(\Phi_j^{(n+1-s)})D^{-1}T_\psi(\Psi_j^{(s)})\,, \tag{A.2}$$

and accordingly for (117):

$$\begin{aligned}\delta_{\mathcal{A}}^{(-n)} T_\phi T_\phi^{-1} + T_\phi \mathcal{M}_{\mathcal{A}}^{(-n)} T_\phi^{-1} &= (T_\phi(\mathcal{M}_{\mathcal{A}}^{(-n)}(\phi) - \delta_{\mathcal{A}}^{(-n)}\phi))D^{-1}\phi^{-1} \\ &\quad + \sum_{a,b=1}^{M+R} \mathcal{A}_{ab}^{(-n)} \sum_{s=1}^{n} T_\phi(\Phi_b^{(-n-1+s)}) D^{-1} T_\phi^{*-1}(\Psi_a^{(-s)}) ,\end{aligned} \tag{A.3}$$

$$\begin{aligned}(\delta_{\mathcal{A}}^{(-n)} T_\psi^{*-1}) T_\psi^* + T_\psi^{*-1} \mathcal{M}_{\mathcal{A}}^{(-n)} T_\psi^* &= -\psi^{-1} D^{-1}(T_\psi((\mathcal{M}_{\mathcal{A}}^{(-n)})^*(\psi) + \delta_{\mathcal{A}}^{(-n)}\psi)) \\ &\quad + \sum_{a,b=1}^{M+R} \mathcal{A}_{ab}^{(-n)} \sum_{s=1}^{n} T_\psi^{*-1}(\Phi_b^{(-n-1+s)}) D^{-1} T_\psi(\Psi_a^{(-s)}) .\end{aligned} \tag{A.4}$$

In order for (adjoint) DB transformations to preserve the additional symmetries, i.e. to preserve the form of $\mathcal{M}_A^{(n)}$ and $\mathcal{M}_{\mathcal{A}}^{(-n)}$ one term on the r.h.s. of Eqs. (A.1)–(A.4) must vanish. The first possibility, as in (241)–(242), is the generating (adjoint) eigenfunctions to transform homogeneously under the additional symmetries:

- For positive-grade symmetry flows:

$$\delta_A^{(n)}\phi = \mathcal{M}_A^{(n)}(\phi) , \qquad \delta_A^{(n)}\psi = -(\mathcal{M}_A^{(n)})^*(\psi) , \tag{A.5}$$

  which is fulfilled for:

$$\phi = L_M(\bar{\varphi}_{a_0}) \equiv \Phi_{a_0}^{(-1)} , \quad \psi = \bar{\psi}_{a_0} \equiv \Psi_{a_0}^{(-1)} , \quad \text{for } a_0 = \text{fixed} , \tag{A.6}$$

  due to (120).
- For the negative-grade symmetry flows:

$$\delta_{\mathcal{A}}^{(-n)}\phi = \mathcal{M}_{\mathcal{A}}^{(-n)}(\phi) , \qquad \delta_{\mathcal{A}}^{(-n)}\psi = -(\mathcal{M}_{\mathcal{A}}^{(-n)})^*(\psi) , \tag{A.7}$$

  which is fulfilled for:

$$\phi = \Phi_{i_0} , \quad \psi = \Psi_{i_0} , \quad \text{for } i_0 = \text{fixed} , \tag{A.8}$$

  due to (118).

The second possibility is one of the terms in the sums on the r.h.s. of (A.1)–(A.4) to vanish, which implies to make the opposite choice w.r.t. (A.6)–(A.8):

- Choose (A.8) for positive-grade symmetries in (A.1)–(A.2)
- Choose (A.6) for negative-grade symmetries in (A.3)–(A.4).

It is easy to find out that these latter choices for (adjoint) DB transformations preserve a subalgebra of the additional loop algebra symmetries (125). Namely, for positive-grade symmetries (where initially $A^{(n)} \in \mathrm{U}(1) \oplus \mathrm{SL}(M)$), we have to restrict the set of matrices $A^{(n)}$ as follows ($i_0 = \text{fixed}$):

$$A_{ji_0}^{(n)} = A_{i_0 j}^{(n)} = \delta_{i_0 j} \rightarrow A^{(n)} \in \mathrm{U}(1) \oplus \mathrm{SL}(M-1) \tag{A.9}$$

for consistency with (adjoint) DB transformations w.r.t. (A.8). For negative-grade symmetries (where initially $\mathcal{A}^{(-n)} \in \mathrm{SL}(M+R)$) we have to restrict the set of matrices $\mathcal{A}^{(-n)}$ as ($a_0 =$ fixed):

$$\mathcal{A}^{(-n)}_{ba_0} = \mathcal{A}^{(-n)}_{a_0 b} = \delta_{a_0 b} \to \mathcal{A}^{(-n)} \in \mathrm{SL}(M+R-1) \tag{A.10}$$

for consistency with (adjoint) DB transformations w.r.t. (A.6).

In conclusion, we obtain:

- (Adjoint) DB transformations w.r.t. (A.6) preserve $(\widehat{\mathrm{U}}(1) \oplus \widehat{\mathrm{SL}}(M-1))_+ \oplus (\widehat{\mathrm{SL}}(M+R))_-$ subalgebra of additional symmetries.
- (Adjoint) DB transformations w.r.t. (A.8) preserve $(\widehat{\mathrm{U}}(1) \oplus \widehat{\mathrm{SL}}(M))_+ \oplus (\widehat{\mathrm{SL}}(M+R-1))_-$ subalgebra of additional symmetries.

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